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Singularities of integrals

Homology, hyperfunctions and microlocal analysis

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Contents

Foreword

Many times throughout the course of their history, theoretical physics and mathematics have been brought together by grand structural ideas which have proved to be a fertile source of inspiration for both subjects.

The importance of the structure of holomorphic functions in several variables became apparent around 1960, with the mathematical formulation of the quantum theory of fields and particles. Indeed, their complex singularities, known as "Landau" singularities, form a whole universe, and their physical interpretation is part of a particle physicist's basic conceptual toolkit. Thus, the presence of a pole in an energy or mass variable indicates the existence of a particle, and singularities of higher complexity are a manifestation of a ubiquitous geometry which underlies classical relativistic multiple collisions and also includes the creation of particles, which lies at the heart of quantum interaction processes.

On the pure mathematics side, one can safely say that this branch of theoretical physics genuinely contributed to the birth of the theory of hyperfunctions and microlocal analysis. (For example, one can mention the initial motivation in the work of M. Sato at the time of the "dispersion relations", which came from physics, and also the work of A. Martineau, B. Malgrange, and M. Zerner on the "edge-of-the-wedge" theorem, which came out of the first meetings in Strasbourg between physicists and mathematicians).

For a mathematical physicist, these holomorphic structures in quantum field theory have a deep meaning, and are inherent in the grand principles of relativistic quantum field theory: Einstein causality, the invariance under the Poincaré group, the positivity of energy, the conservation of probability or "unitarity", and so on. But it is in the "perturbative" approach to quantum field theory (whose relationship with "complete" or "non-perturbative" theory can be compared to the relationship between the study of formal power series and convergent series), that the holomorphic structures which generate Landau singularities appear in their most elementary form: namely, as holomorphic functions defined by integrals of rational functions associated to "Feynman diagrams".

It is to Frédéric Pham's great credit that he undertook a systematic analysis of these mathematical structures, whilst a young physicist at the Service de Physique Théorique at Saclay, using the calculus of residues in several variables as developed by J. Leray, together with R. Thom's isotopy theorem. This fundamental study of the singularities of integrals lies at the interface between analysis and algebraic geometry, and culminated in the Picard–Lefschetz formulae. It was first published in 1967 in the Mémorial des Sciences Mathématiques (edited by Gauthier-Villars), and was subsequently followed by a second piece of work in 1974 (the content of a course given at Hanoi), where the same structures, enriched by the work of Nilsson, were approached using methods from the theory of differential equations and were generalized from the point of view of hyperfunction theory and microlocal analysis.

It seemed to us that, because of the importance of their content and the wide range of different approaches that are adopted, a new edition of these texts could play an extremely important role not only for mathematicians, but also for theoretical physicists, given that the major fundamental problems in the quantum theory of fields and particles still remain unsolved to this day.

First of all, we note that the methods developed by Frédéric Pham have had wide-ranging impact in the non-perturbative approach to this subject, by allowing one to study holomorphic solutions to integral equations in complex varieties with varying cycles. Such equations are inherent in the general formalism of quantum field theory (they are known as equations of "Bethe–Salpeter" type, and are intimately connected to general unitarity relations). In this way, it is possible, for example, in the case of collisions between massive particles which are described in a general manner by the structure functions of a quantum field theory, to disentangle the singularities of a "three-particle threshold", which appears as an accumulation point of "holonomic singularities" (here we refer to a classification of the singularities of collision amplitudes considered by M. Sato).

Some extremely tough problems remain open concerning this type of structure in the case when massless particles are involved in collisions, which is of considerable physical importance. The type of analysis that we have just described above (the case of collisions involving only massive particles) concerns properties of field theory which are independent of "problems of renormalization", which are a different category of crucially important problems. These appear on the perturbative level as the need to redefine primitively-divergent Feynman integrals, and have recently been cast in a new light by the work of Connes and Kreimer, who brought a Hopf algebra structure into the picture. Nonetheless, on a general, non-perturbative level, the problems of renormalization still appear to be at the very source of the problem of the existence of non-trivial field theories in four-dimensional space-time.

This "existential problem" was brought to light by Landau in 1960 in the resummation of renormalized perturbation series in quantum electrodynamics and is exhibited, at a more general level, by the simplest scalar field theory with quartic interaction term. This phenomenon involves the "generic" creation of poles (and other Landau singularities) as a result of renormalization, in a region of complex space where such singularities are forbidden by general principles. *In fact, the existential flaw of these models appears to be related in a deep way to the absence of an important property, known as "asymptotic freedom", for the field structure functions at very high energies.*

Are non-abelian gauge theories, which are conceptually richer and also closer to the experimental complexity of particle physics, capable – as one could hope since they can incorporate asymptotic freedom – of reconciling renormalization with the fundamental holomorphic structures which arise from the grand principles of relativistic quantum physics? In the current state of particle physics, in which the "standard model" of field theory is considered to be very reliable in a vast range of energies, the construction and mathematical analysis of the simplest quantum field theory with non-abelian gauge, namely a Yang–Mills model, is a challenge which cannot be avoided (and the reason why it has been chosen as the subject of one of the major mathematical problems posed by the "Clay Institute" in the year 2000 ...).

In such a conceptually rich subject as quantum relativistic field theory, it seems extremely important to continue to develop the point of view of holomorphic structures and complex singularities in tandem with structures such as symmetries and gauge groups, which have been the most recent driving forces for research. In this light, the present work offers us a panoply of results from which a mathematical physicist should be able to draw great benefit.

Jacques Bros
(*Service de Physique Théorique, CEA Saclay*)

Part I

Introduction to a topological study of Landau singularities

Introduction

This work is an up-to-date version of a collaboration undertaken by D. Fotiadi, M. Froissart, J. Lascoux and the author, which led to the two articles [12, 29],[1] and whose results are reproduced here and extended (without, however, reproducing all the proofs). I have endeavoured to present a coherent whole which is understandable to the non-mathematical reader, and many sections (and almost the entirety of the first two chapters) merely consist of reminders of well-known mathematical concepts. However, we will assume that the reader is familiar with the basic rudiments of general Topology.[2]

The starting point for this work was the desire to understand an important chapter in the physics of elementary interactions – the problem of "Landau singularities". After reading the numerous works inspired by the article of L.D. Landau [18], one is struck, on the one hand, by the elegance of certain results, such as the "Cutkosky rules" [5], and the universal nature of these results which go beyond the original scope of perturbation theory (as indeed Landau had predicted), and on the other hand, by the extreme complexity which seems to produce a new "pathology" every time the subject is studied in greater depth.

We shall see that a systematic study of the problem can be undertaken in a very general mathematical setting, where the "good" results cease to be miraculous and the "bad" results cease to be pathological. The general problem is to study the analytic properties of functions defined by multiple integrals. The integration can take place in Euclidean space, as in the case of "Feynman integrals" in perturbation theory, or on a submanifold of Euclidean space, as in the case of "unitarity integrals". The integrand is an analytic function which depends analytically on external complex parameters, and by introducing the "associated complex manifold" of the integration manifold, we can assume that the singular locus of the integrand is an *analytic subset* of

[1] The numbers in square brackets [ ] refer to the "Bibliography" on page 143.

[2] Such as can be found at the beginning of the short book by A.H. Wallace [39], for example.

the complex manifold of *all* the parameters (both internal and external). This analytic subset naturally projects onto the manifold of external parameters, and one can consider its *apparent contour*. We shall see that, due to “Thom's isotopy theorem” (chap. IV), the integral is an analytic function outside this apparent contour, which is called the “Landau singularity” of the integral.

Here, we already see an explanation for the “pathologies” observed by physicists: we know, for example, that apparent contour curves in the plane have “generic” cusps. In three-dimensional space, apparent contour surfaces generically have, not only “lines of cusps”, but also isolated singularities called “swallowtails”, and so on.

We have at our disposal a lot of information about singularities which appear “generically” along apparent contours, due especially to the work of R. Thom [36], of which will give only the most superficial account.

Note that we are concerned not simply with apparent contours on manifolds, but on *apparent contours on analytic sets.* To define them, we will use a “*stratification*” of analytic sets, which was a concept introduced by H. Whitney [42] and R. Thom [37]. This is a partition of an analytic set into manifolds known as “strata”. The “incidence relations” between the different strata will form the basis of a classification of Landau singularities, related to what the physicists call the “hierarchy of singularities”.

After these purely geometric considerations, we will describe the nature of the singularities of the analytic function defined by an integral, in the case where the singular locus of the integrand is a union of subvarieties in general position. We will show (in chapters V and VII) how the integration cycle deforms when the external parameters describe a small loop around a Landau singularity; this is the problem of the “ramification of the integral”. From this, we will deduce (in chap. VI) some precise expressions for the “singular part” of the analytic function on its Landau singularity, at least with the hypothesis that the integrand has only polar singularities. We will see that in this case, the integrand can have only poles, algebraic singularities of order two, or logarithmic singularities. Note that this work did not require much creativity, since the essential bulk of it was already done by J. Leray [20].

We will notice in passing that the “Cutkosky rules” are a trivial consequence of the “ramification” formulae given above, and of *Leray's theory of residues*. Several physicists (notably, Polkinghorne's group [13]) had already suspected a link between these Cutkosky rules and the residue calculus, but they were not able to formulate this precisely owing to a lack of adequate mathematical tools. We recall that “homological” concepts, which are essential for Leray's theory, were precisely invented by Poincaré [30] (1895) in order to generalize the residue calculus to many variables, and to compensate for our lack of geometric intuition in “hyperspace”.

Our account will be purely mathematical, but even if it is of clear interest to physicists, applying it in practice is never free of certain problems. I will outline these below.

First of all, it is "unfortunately" easy to see that in certain physical situations, the geometry of the Landau singularities is not "generic" in the sense of Thom. I will broach this question in my thesis (in preparation). Let us only say here that these "accidents", which are not mysterious in any way, do not seem to create any serious difficulties.

Another practical problem is to construct an effective "stratification" of the analytic set under consideration. To the physicist, this subset is simply a union of submanifolds in general position in affine Euclidean space, and so the stratification is trivial. But unfortunately, we are only allowed to apply Thom's isotopy theorem to the "*compactified*" set,[3] which poses the apparently rather difficult problem of constructing a stratification "at infinity". It would be important to know how to solve this in order to exhibit all "Landau singularities of the second kind" (which is the name by which the apparent contours of the "strata at infinity" are known to the physicists [9]).

Let us also point out that the fundamental mathematical problem is to extend the results obtained here, which are essentially local, to *global* results, and this is the main point of using homological concepts. We can, for example, ask the following question: are all "Landau singularities" effectively singular "somewhere" (in other words, on at least one "branch" of the multivalued function defined by the integral)? This leads to the following homological problem. Chapter V (§1.2) defines a representation of the fundamental group of the "base" (the manifold of the external parameters) in the homology space of the "fibre" (the integration manifold). How can one know if this representation is *irreducible*? It goes without saying that the "recipe" proposed at the beginning of section 3 (in chap. V) for solving such problems, is merely wishful thinking, since it is only easy to apply for the simplest of Feynman integrals (cf. an unpublished article by D. Fotiadi and the author). For a slightly less trivial Feynman integral, the calculation of the homology groups is in its own right a difficult problem (cf. a paper by P. Federbush [11]). One should recognize the fact that the general setting in which we have cast the problem is a little too simplistic, since we have mainly used the differential structure of the manifolds being studied, and a little of their analytic structure, but we have not at all used their *algebraic structure*.[4]

I wish to express my thanks to D. Fotiadi, M. Froissart, and J. Lascoux, for the many discussions out of which this work grew; to Professor J. Leray whose "theory of residues" provided us with the initial ideas; and to Professor R. Thom for the patience and kindness with which he guided us.

[3] Not to mention the need to have a compact domain of integration in order for the integral to make sense.

[4] Let us indicate two recent articles where algebraic concepts come into play. In [32], Regge and Barucchi study some Landau curves using methods from algebraic geometry, and in [24], Nilsson defined a class of multivalued analytic functions with certain growth properties whose singular locus is an algebraic subset, and showed that this class is stable under integration.

I

Differentiable manifolds

1 Definition of a topological manifold

An *n-dimensional manifold* is a Hausdorff (1)[1] topological space X such that:

(X_0) Every point $x \in X$ has a neighbourhood U_x, which is homeomorphic to an open subset E_x of Euclidean space $\mathbb{R}^n$;

(X_1) X can be covered by at most a *countable* number of such neighbourhoods.

A homeomorphism $h : U \xrightarrow{\approx} E$ is called a *chart* on the neighbourhood U, or "*local chart*" on X, and U is called the "*domain*" of the local chart h.

We shall always assume from now on that these "domains" U are open connected sets (this convention is purely for convenience, and does not result in any loss of generality).

A family of local charts $\left\{h_i : U_i \xrightarrow{\approx} E_i\right\}$ such that $\bigcup_i U_i = X$ is called an *atlas* of X.

Property (X_1) can therefore be stated as follows: "X has a countable atlas".

2 Structures on a manifold

2.1

To equip the manifold X with a "*differentiable structure* " is to require that all the local charts on X are "patched together" differentiably: we say that two charts $h : U \to E$, $h' : U' \to E'$ "can be patched together differentiably" if the map $h' \circ h^{-1}$, defined on the open set $h(U \cap U')$, is differentiable (Fig. I.1).

In the same way, one can define a $\mathscr{C}^r$ differentiable structure (using functions which are r times continuously differentiable), or an analytic structure, and so on. If the manifold X is of even dimension $2n$, we will define the notion

[1] Numbers such as (1) refer to "Technical notes" on page 137.

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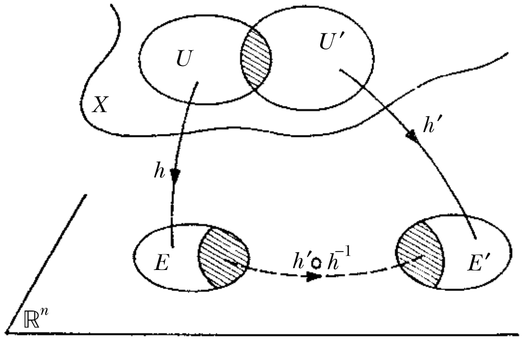

Fig. I.1.

of "complex analytic structure" by identifying real Euclidean space $\mathbb{R}^{2n}$ with complex Euclidean space $\mathbb{C}^n$:

$$(\xi_1, \xi_2, \ldots, \xi_{2n-1}, \xi_{2n}) \rightsquigarrow (\xi_1 + i\xi_2, \ldots, \xi_{2n-1} + i\xi_{2n}) \,,$$

and by requiring the map $h' \circ h^{-1}$ to be complex analytic. In this case we say that n is the "complex dimension" of the variety X.

Let us recap our definition of a differentiable structure (for the sake of argument) in greater detail.

A "*differentiable atlas*" is an atlas in which all charts are patched together differentiably; we say that two differentiable atlases are "*equivalent*" if their union is a differentiable atlas; and *a differentiable structure on the manifold X* is then a *differentiable atlas* on it, considered *up to equivalence.*

2.2 Morphisms of manifolds

Let X and Y be two manifolds of dimension n and p respectively, each equipped with a differentiable structure (for example). A continuous map $f : X \to Y$ is called a "*differentiable map*" (or a "morphism between the differentiable structures on X and Y") if for every local chart $h : U \to E$ on X, and for every local chart $k : V \to F$ on Y (which are compatible with the given differentiable structures) such that $f(U) \subset V$, the map $k \circ f \circ h^{-1} : E \to F$ is differentiable (Fig. I.2).

If, moreover, the inverse map f^{-1} exists and is also a "differentiable map", then we say that f is an *isomorphism between differentiable manifolds*[2] (and, in this case, we necessarily have $n = p$).

Special case. Y is the real line $\mathbb{R}$, with its obvious differentiable structure. A morphism $f : X \to \mathbb{R}$ is then called a *differentiable function* on X.

[2] In particular, a chart $h : U \to E$ is an *isomorphism* between the differentiable manifolds U and E.

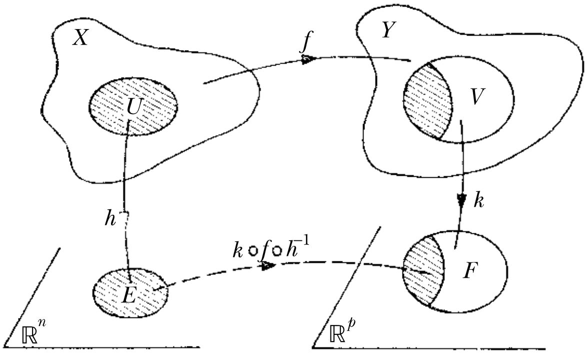

Fig. I.2.

2.3 Remark. By "differentiable manifold", we mean a manifold X equipped with a differentiable structure $\mathscr{S}$. A morphism of differentiable manifolds should therefore be written $f : (X, \mathscr{S}) \to (Y, \mathscr{S}')$, specifying the differentiable structures $\mathscr{S}$ and $\mathscr{S}'$ attached to X and Y, *because the same space can have several differentiable structures.*[3] We will almost always omit this extra information when no confusion can arise.

2.4 Examples of differentiable manifolds

(i) Euclidean space $\mathbb{R}^n$, or an open subset $E \subset \mathbb{R}^n$, carry the obvious differentiable structure, with an atlas consisting of a single chart, namely the identity map $\mathbb{1}_E : E \to E$.

(ii) A slightly less trivial example is *projective space* P^n, the set of directions of lines in Euclidean space $\mathbb{R}^{n+1}$: two points x, $x' \in \mathbb{R}^{n+1} - \{0\}$ define the same point $\overline{x} \in P^n$ if there exists a non-zero scalar λ such that $x' = \lambda x$. The coordinates $x_0, x_1, \ldots, x_n$ of the point $x \in \mathbb{R}^{n+1} - \{0\}$ are called the homogeneous coordinates of the point $\overline{x} \in P^n$. Let us denote by U_i the open subset (2) of P^n given by the set of points whose i-th homogeneous coordinate is non-zero:

$$U_i = \{\overline{x} : x_i \neq 0\} \qquad (i = 0, 1, 2, \ldots, n).$$

[3] Let X be a manifold, let $\mathscr{S}$ be a differentiable structure on X, and let $f : X \to X$ be a homeomorphism which is *non-differentiable* (for the structure $\mathscr{S}$). Such a homeomorphism transforms the atlases which define the structure $\mathscr{S}$ into non-equivalent atlases, which will therefore define a *different structure* $f\mathscr{S}$. Clearly, this is not very interesting, because the new structure is homeomorphic to the previous one [$f : (X, \mathscr{S}) \to (X, f\mathscr{S})$ is, by construction, an isomorphism]. But J. Milnor [23] gave an example of a manifold (the seven-dimensional sphere) equipped with *two non-isomorphic differentiable structures.*

This open set, which is obviously connected, can be equipped with a chart $h_i : U_i \to \mathbb{R}^n$ defined in the following way: h_i maps the point $\overline{x} = (x_0, x_1, \ldots, x_n)$ to the point $\xi \in \mathbb{R}^n$ with coordinates

$$\xi_1 = \frac{x_0}{x_i}, \quad \xi_2 = \frac{x_1}{x_i}, \quad \ldots, \quad \xi_i = \frac{x_{i-1}}{x_i}, \\ \xi_{i+1} = \frac{x_{i+1}}{x_i}, \quad \ldots, \quad \xi_n = \frac{x_n}{x_i}.$$

The maps h_i are obviously homeomorphisms, and form an atlas on P^n. Let us check that this atlas defines an *analytic structure* on P^n. Let $\overline{x} \in U_i \cap U_j$ $(i > j)$ and $\xi = h_j(\overline{x})$. The point $\overline{x}$ can be represented by the homogeneous coordinates $(\xi_1, \xi_2, \ldots, \xi_{j-1}, 1, \xi_j, \ldots, \xi_n)$, and the point $h_i(\overline{x}) = h_i \circ h_j^{-1}$ has coordinates

$$\left(\frac{\xi_1}{\xi_i}, \frac{\xi_2}{\xi_i}, \ldots, \frac{\xi_{j-1}}{\xi_i}, \frac{1}{\xi_i}, \frac{\xi_j}{\xi_i}, \ldots, \frac{\xi_{i-1}}{\xi_i}, \frac{\xi_{i+1}}{\xi_i}, \ldots, \frac{\xi_n}{\xi_i}\right);$$

therefore the transformation $h_i \circ h_j^{-1}$ is indeed analytic on U_i.

(iii) *Complex projective space* $\mathbb{C}P^n$ is defined in the same way: two points $x, x' \in \mathbb{C}^{n+1} - \{0\}$ define the same point $\overline{x} \in \mathbb{C}P^n$ if there exists a scalar $\lambda \in \mathbb{C} - \{0\}$ such that $x' = \lambda x$.
The same argument as above shows that $\mathbb{C}P^n$ is a complex analytic manifold.
This manifold is *compact* (as is P^n): to see this, let

$$K = \{\xi \in \mathbb{C}^n : |\xi_1| \leqslant 1, \ldots, |\xi_n| \leqslant 1\}$$

be the (compact) unit "polydisk" in Euclidean space. One sees immediately that $\mathbb{C}P^n$ is *covered* by the *finite* family of *compact* sets $K_i = h_i^{-1}(K) \subset U_i$, which are the inverse images of K by the charts h_i, defined above.

3 Submanifolds

3.1

Let X be an n-dimensional manifold, equipped with a differentiable structure, for the sake of argument.

Suppose we are given a subspace S of X, and suppose that the following property is satisfied: there exists a family $\{h_i : U_i \to E_i\}$, which is compatible with the differentiable structure on X, and local charts on X, whose domains U_i cover $S : \bigcup_i U_i \supset S$, and which send S into a p-dimensional plane $\mathbb{R}^p \subset \mathbb{R}^n$:

$$h_i(S \cap U_i) = \mathbb{R}^p \cap E_i = \{\xi \in E_i : \xi_{p+1} = \cdots = \xi_n = 0\} \quad \text{(cf. Fig. I.3)}.$$

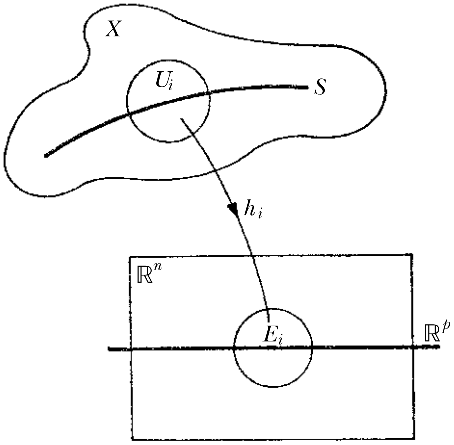

Fig. I.3.

It is clear that the restriction of these local charts to S defines a *differentiable atlas* of S and that the subspace S is thus equipped with the structure of a *p-dimensional differentiable manifold*, which depends only on the structure on X (and not on the chosen family $\{h_i\}$). We say that S is a *p-dimensional differentiable submanifold* (of codimension $n-p$) of X.

Special case. A submanifold of codimension 0 is simply an open subset of X.

3.2 Embeddings

If S is a submanifold of Y, the inclusion $i : S \overset{\subset}{\rightarrow} Y$ (which to every point $y \in S$ associates the same point $y \in Y$) is obviously a *morphism*. Such a morphism is called an "*embedding*". More generally, an *embedding* of a manifold X into the manifold Y will be any morphism $f : X \to Y$ which factorizes in the following way

$$f : X \underset{\approx}{\xrightarrow{h}} S \underset{\subset}{\xrightarrow{i}} Y,$$

where h is an *isomorphism* of X onto the submanifold S of Y.

3.3 Immersions

A local embedding will be called an *immersion*. In other words, an immersion is a morphism $f : X \to Y$ such that every point $x \in X$ has a neighbourhood U_x for which

$$f|U_x : \ U_x \longrightarrow Y$$

is an embedding.

3.4 Examples. The following examples are given without any proofs. In fact, they will be justified by §§4.4, 4.5, 4.6.

(i) The set S of points in Euclidean space $\mathbb{R}^n$ which satisfy the equation $s(x) = 0$, where s is a differentiable function whose gradient never vanishes at the same time as s, is a *closed differentiable submanifold* of $\mathbb{R}^n$.
The intersection of this submanifold with an open subset of $\mathbb{R}^n$ is again a differentiable submanifold of $\mathbb{R}^n$ (which is not closed).

(ii) The map $f : P^{n-1} \to P^n$ which sends the point with homogeneous coordinates $(x_0, x_1, \ldots, x_{n-1})$ in P^{n-1} to the point with homogeneous coordinates $(x_0, x_1, \ldots, x_{n-1}, 0)$ in P^n is an *embedding*.

(iii) A plane unicursal curve, given by the parametrization $x_1 = f_1(t)$, $x_2 = f_2(t)$, where f_1 and f_2 are two differentiable functions whose derivatives do not vanish simultaneously, defines an *immersion*

$$f : \mathbb{R} \longrightarrow \mathbb{R}^2.$$

This immersion is in general not an embedding (for example, the curve defined by the image can have double points).

(iv) Let $f : \mathbb{R} \to \mathbb{R}^2$ be the map defined by $x_1 = t^2$, $x_2 = t^3$. This analytic map, even though it is injective, is not an embedding, nor even an immersion (Fig. I.4).

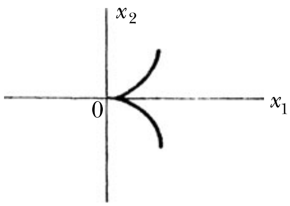

Fig. I.4.

4 The tangent space of a differentiable manifold

4.1 The tangent vector space $T_x(X)$

If X is an n-dimensional differentiable manifold, an *n-dimensional vector space* $T_x(X)$ is associated to every point $x \in X$ in the following way.

An element $V \in T_x(X)$, called a *tangent vector*, is defined by specifying a point x and a pair (h, v), where h is a chart on a neighbourhood of x, and v is a vector in $\mathbb{R}^n$. We say that two pairs (h, v), (h', v') define the same tangent vector V if

$$v' = T_\xi(h' \circ h^{-1})v,$$

where $T_\xi(h' \circ h^{-1})$ is the Jacobian matrix of the map $h' \circ h^{-1}$ calculated at point $\xi = h(x)$. The coordinates $v_1, v_2, \ldots, v_n$ of the vector v are called the "*components of the vector V in the chart h*" and the above relation is the "change of chart rule" for tangent vectors.

If φ is a differentiable function

$$\varphi : X \longrightarrow \mathbb{R}$$

and V is a tangent vector to X at the point x, one easily verifies that the number

$$V \cdot \varphi = \sum_{i=1}^{n} v_i \frac{\partial \left(\varphi \circ h^{-1}\right)(\xi)}{\partial \xi_i} \qquad [\xi = h(x)]$$

is independent of the pair (h, v) which defines the vector V. It is called the "*derivative of φ in the direction V*".

Remark. As this interpretation of tangent vectors as *derivations* suggests, the following notation is frequently used. Given a chart h, we let $\partial/\partial\xi_i$ denote the tangent vector whose coordinates in the chart h are all zero except for the i-th coordinate, which is 1.

4.2 The tangent map

Let $f : X \to Y$ be a differentiable map from an n-dimensional manifold X to a p-dimensional manifold Y.

The tangent map

$$T_x f : T_x(X) \longrightarrow T_y(Y) \quad [y = f(x)]$$

is the linear map defined by the rule

$$V = (h, v) \rightsquigarrow W = (k, w);$$
$$w = T_\xi(k \circ f \circ h^{-1})v \quad [\xi = h(x)]\,,$$

which is obviously compatible with the change of chart rule for tangent vectors.

This rule is also compatible with the interpretation of tangent vectors as *derivations*: every differentiable function $\varphi : Y \to \mathbb{R}$ defines, by composing with f, a differentiable function $\varphi \circ f : X \to \mathbb{R}$, and we have

$$(T_x\, f\, V) \cdot \varphi = V \cdot (\varphi \circ f)\,.$$

Even though the following two properties are obvious, they are worth stating because we will see them again and again in the sequel:

1° The identity map $\mathbb{1}_X : X \to X$ corresponds to the identity map

$$T_x(\mathbb{1}_x) = \mathbb{1}_{T_x(X)} : T_x(X) \longrightarrow T_x(X);$$

2° The composition of two maps

$$g \circ f : \begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ x & & y & & z \end{array}$$

corresponds to the composition of their tangent maps

$$T_x(g \circ f) = (T_y g) \circ (T_x f) : T_x(X) \xrightarrow{T_x f} T_y(Y) \xrightarrow{T_y g} T_z(Z).$$

We can summarize these two properties by saying that the correspondence $f \rightsquigarrow T_x f$ is a "*covariant functor*".

4.3 Definition. The *rank of the map f at the point x*, denoted $\mathrm{rank}_x f$, is the rank of the tangent map $T_x f$.

4.4 Characterization of an immersion. *A differentiable map $f : X^n \to Y^p$ is an immersion if and only if it has rank n everywhere ($n \leqslant p$).*

The notion of immersion, like that of the rank of a map, is local. One can therefore reduce (using local charts on X and Y) to the case where X and Y are open subsets of Euclidean space:

$$X = E^n \subset \mathbb{R}^n, \quad Y = E^p \subset \mathbb{R}^p.$$

The map $f : E^n \to E^p$ is therefore given by p functions of n variables $f_1(x), \ldots, f_p(x)$, and we will assume that E^n and E^p contain the coordinate origin, with $f(0) = 0$.

If f is of rank n in E^n, we can assume, for example, that the upper n by n minor in the table

$$\begin{array}{c|c} {}_j\diagdown^i & 1, \ldots, n \\ \hline 1 & \\ \vdots & \dfrac{\partial f_j}{\partial x_i} \\ n & \\ \vdots & \\ p & \end{array}$$

is non-zero in a neighbourhood of 0.

Therefore the functions

$$g_j(y_1, \ldots, y_p) = \begin{cases} f_j(y_1, \ldots, y_n) & (j \leqslant n), \\ f_j(y_1, \ldots, y_n) + y_j & (j > n) \end{cases}$$

locally define a differentiable map g from $\mathbb{R}^p$ to $\mathbb{R}^p$, whose tangent map is represented by the matrix

$$\begin{array}{c|c|c} {}_{j}\diagdown^{i} & 1,\ \ldots & n,\ldots,p \\ \hline \begin{matrix}1\\ \vdots \\ n\end{matrix} & \dfrac{\partial f_j}{\partial x_i} & 0 \\ \hdashline \begin{matrix}\vdots \\ p\end{matrix} & & \mathbb{1} \end{array}$$

and whose determinant is non-zero.

Therefore, by the implicit function theorem, the (differentiable) inverse map g^{-1} exists in a neighbourhood of 0, and we have an *isomorphism*

$$g : U^p \xrightarrow{\approx} U'^p \qquad (U^p, U'^p \subset E^p)\,.$$

But g coincides with f on the n-plane $\mathbb{R}^n \subset \mathbb{R}^p$ defined by the equations $y_j = 0$, $j > n$, so that the *isomorphism* g sends the *n-plane* $U^n = \mathbb{R}^n \cap U^p$ onto the set $S = f(U^n)$. This set is therefore a *submanifold of* E^p, according to the definition of §3.1, which means that f is an *immersion*.

The converse is obvious (if f is an immersion, it has rank n everywhere).

4.5 Submersions. A differentiable map $f : X^n \to Y^p$ whose rank is everywhere equal to p $(n \geqslant p)$ is called a "submersion".

Proposition. *If f is a submersion, then for every $y \in Y$, the set $f^{-1}(y)$ is a submanifold of X of codimension p.*

In this case also, we can reduce to the case of Euclidean spaces. If the p by p minor on the left of the following table

$$\begin{array}{c|c:c} {}_{j}\diagdown^{i} & 1,\ \ldots & p,\ldots,n \\ \hline \begin{matrix}1\\ \vdots \\ p\end{matrix} & \dfrac{\partial f_j}{\partial x_i} & \end{array}$$

is non-zero, the functions

$$g_j(x) = \begin{cases} f_j(x) & (j \leqslant p), \\ x_j & (j > p) \end{cases}$$

define a map g from $\mathbb{R}^n$ to $\mathbb{R}^n$, whose tangent map is represented by the matrix

$$\begin{array}{c|cc} {}_{j}\diagdown^{i} & 1,\ \ldots & p,\ldots,n \\ \hline 1 & & \\ \vdots & \dfrac{\partial f_j}{\partial x_i} & \\ p & & \\ \vdots & 0 & \mathbb{1} \\ n & & \end{array}$$

which has non-vanishing determinant.

It follows from the implicit function theorem that g defines an *isomorphism* $g : U^n \to U'^n$. This isomorphism sends the set $S_0 = U^n \cap f^{-1}(0)$ onto the $(n-p)$-plane $U^{n-p} = \mathbb{R}^{n-p} \cap U'^n$ defined by the equations $x_1 = \cdots = x_p = 0$. The set S_0 is therefore a submanifold of codimension p.

4.6 Characterization of a submanifold by local equations. *For $S \subset X$ to be a submanifold of codimension q, it is necessary and sufficient that each point $y \in S$ have a neighbourhood U_y in X for which*

$$S \cap U_y = \{x \in U_y : s_{1y}(x) = \cdots = s_{qy}(x) = 0\},$$

where $\{s_{1y}, \ldots, s_{qy}\}$ is a system of q functions defined on U_y which has rank q (in other words, it defines a map $s_y : U_y \to \mathbb{R}^q$ of rank q).

It is clear that the condition is necessary, because if S is a submanifold, the functions $\xi_{p+1}, \ldots, \xi_n$ corresponding to the chart h_i of §3.1 form a system of rank $q = n - p$ in U_i.

The converse is nothing but a restatement of the proposition in §4.5.

Definitions. We will say that $s_{1y}, \ldots, s_{qy}$ are the *local equations* of the submanifold S in the neighbourhood of the point y.

If there exists a neighbourhood V which *covers the whole of* S, and if there exist functions $s_1, \ldots, s_q$ defined on V such that

$$S = \{x \in V : s_1(x) = \cdots = s_q(x) = 0\},$$

then we will say that these functions are *global equations* of S.

Examples.

(i) The submanifold $P^{n-1} \subset P^n$ of example 3.4 (ii) has a local equation given by the function x_n/x_i [defined on the open set U_i, of §2.4 (ii)]: one would like to say that P^{n-1} has x_n as its global equation, but note that the homogeneous coordinate x_n is not well-defined as a function on P^n.

(ii) Let S^n be the n-dimensional sphere in Euclidean space $\mathbb{R}^{n+1}$. Its intersection with a hyperplane (say, $x_1 = 0$) is a submanifold $S^{n-1} \subset S^n$, which has the *global* equation x_1 (defined on the whole of S^n).

Remark. Since the implicit function theorem also exists for *analytic* maps, the results of §§4.4, 4.5, 4.6 also apply to analytic structures.

5 Differential forms on a manifold

5.1 The vector space $\Omega_x^p(X)$ of forms of degree p

Let X be a differentiable manifold of dimension n, and let us choose a point $x \in X$ once and for all.

A *form of degree p*

$$\varphi \in \Omega_x^p(X)$$

is a *multilinear antisymmetric function* of p tangent vectors $V_1, V_2, \ldots, V_p \in T_x(X)$.

By multilinearity and antisymmetry,

$$\varphi\,(V_1, V_2, \ldots, V_p) = 0$$

whenever the vectors $V_1, V_2, \ldots, V_p$ are linearly dependent. In particular, there is no non-zero form of degree greater than n.

The $\Omega_x^p(X)$ are clearly vector spaces. We can equip these vector spaces with an "*exterior product*".

$$\wedge : \Omega_x^p(X) \otimes \Omega_x^q(X) \longrightarrow \Omega_x^{p+q}(X)$$

defined by

$$(\varphi \wedge \psi)(V_1, V_2, \ldots, V_{p+q}) = \frac{1}{(p+q)!} \sum_i (-)^i \varphi(V_{i_1}, \ldots, V_{i_p}) \psi(V_{i_{p+1}}, \ldots, V_{i_{p+q}})$$

where the sum is over the set of permutations i of $\{1, 2, \ldots, p+q\}$, and $(-)^i$ is the signature of the permutation.

It follows immediately from this definition that

$$\varphi \wedge \psi = (-)^{pq} \psi \wedge \varphi.$$

The vector spaces $\Omega_x^p(X)$ are finite-dimensional. Let h be a chart on a neighbourhood of x, and let

$$\frac{\partial}{\partial \xi_1}, \quad \frac{\partial}{\partial \xi_2}, \quad \ldots, \quad \frac{\partial}{\partial \xi_n}$$

denote the vectors in $T_x(X)$ which are represented in this chart by the unit vectors along the coordinate axes in $\mathbb{R}^n$. Let

$$d\xi_i \in \Omega_x^1(X) \quad (i = 1, 2, \ldots, n)$$

denote the form of degree 1 defined by

$$d\xi_i\left(\frac{\partial}{\partial \xi_i}\right) = \delta_{ij} \quad \text{(the Kronecker symbol).}$$

Then one immediately verifies that a *basis* of $\Omega_x^P(X)$ is given by the forms

$$d\xi_{i_1} \wedge d\xi_{i_2} \wedge \cdots \wedge d\xi_{i_p} \quad (i_1 < i_2 < \cdots < i_p).$$

In the case $p = 0$, we set $\Omega_x^0(X) = \mathbb{R}$, and the exterior product

$$\wedge : \Omega_x^0(X) \otimes \Omega_x^p(X) \longrightarrow \Omega_x^p(X)$$

is simply multiplication by a scalar in the vector space $\Omega_x^q(X)$.

5.2 The space $\Omega^p(X)$ of differential forms of degree p

In everything that follows, the word "differentiable" means "*differentiable of class* $\mathscr{C}^\infty$".

Suppose we are given a form $\varphi(x) \in \Omega_x^p(X)$, for each point $x \in X$, *which varies differentiably as a function of* x: in other words, in every chart h, the coefficients of the expansion of $\varphi(x)$ as a sum of exterior products of $d\xi_i(x)$ are differentiable functions of x. We then say that this defines a *differential form of degree* p: $\varphi \in \Omega^p(X)$.

In addition to the "*exterior product*" of §5.1, the spaces $\Omega^p(X)$ are also equipped with an "*exterior differential*"

$$d : \Omega^p(X) \longrightarrow \Omega^{p+1}(X)$$

defined by

$$(d\varphi)(V_0, V_1, \ldots, V_p) = \sum_{i=0}^{p} (-)^i V_i \cdot \varphi(V_0, \ldots, \widehat{V_i}, \ldots, V_p),$$

where the derivative of a form in the direction V_i is defined by differentiating the coefficients of its expansion in a chart h.

In particular, if $p = 0$, φ is simply a *differentiable function on* X, and $d\varphi$ is the differential form of degree 1 which maps the vector V_0 to the derivative of φ in the direction V_0: this justifies the notation $d\xi_i$ adopted at the end of §5.1 if $\xi_1, \xi_2, \ldots, \xi_n$ are the functions which define the chart $h : X \to \mathbb{R}^n$.

Properties of the exterior differential.

1° $dd = 0$ ("Poincaré's theorem"): this follows from the definition, using the fact that
$$\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_i}.$$

2° $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-)^p \varphi \wedge d\psi$, where $p =$ the degree of φ.

5.3 Transformations of differential forms

Every differentiable map

$$f : X \longrightarrow Y$$

induces a map

$$f^* : \Omega^p(Y) \longrightarrow \Omega^p(X)$$

defined by the "transpose" of the tangent map, in other words

$$[f^* \varphi]\,(V_1, \ldots, V_p)_x = \varphi\,(T_x\, f\, V_1, \ldots, T_x\, f\, V_p)_y$$

[the indices x and $y = f(x)$ have been added to indicate the points where the tangent vectors V_i, $T_x\, f\, V_i$ are defined].

This map f^* has the fundamental property of *commuting with the exterior differential*:

$$\boxed{f^* d = d\, f^*}$$

(to see this, use §4.2).

Furthermore, it follows from the "functorial" properties of §4.2 that:

1° $\mathbb{1}_x^* = \mathbb{1}_{\Omega_p(x)}$;

2° if $g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$, then

$$(g \circ f)^* = f^* g^* : \Omega^p(Z) \xrightarrow{g^*} \Omega^p(Y) \xrightarrow{f^*} \Omega^p(X).$$

We can summarize these two properties by saying that the map $f \rightsquigarrow f^*$ is a "*covariant functor*".

Special cases.

(i) If $p = 0$, φ is a function, and $f^* \varphi = \varphi \circ f$.

(ii) Let S be a submanifold of X, $i : S \to X$ the inclusion map. If $\varphi \in \Omega^p(X)$, then $i^* \varphi \in \Omega^p(S)$ is called the *restriction of* φ *to* S, and is denoted by $\varphi | S$.

5.4 Complex-valued differential forms

Instead of defining forms which take their values in $\mathbb{R}$, as we did in §5.1, we could obviously have defined them to take values in $\mathbb{C}$. In general, this is what is done when X is a complex analytic manifold: a *differential form* on a complex analytic manifold X (of complex dimension n), means a complex-valued differential form defined on the real $2n$-dimensional submanifold underlying X. In a local chart, such a form can be expanded in terms of exterior products of $d\xi_1, d\eta_1, \ldots, d\xi_n, d\eta_n$, where $\xi_1, \eta_1, \ldots, \xi_n, \eta_n$ are the coordinates of the $\mathbb{R}^{2n}$ underlying $\mathbb{C}^n$. In practice, it is often more convenient to expand it in terms of the forms

$$d\zeta_j = d\xi_j + i d\eta_j \quad \text{and} \quad d\overline{\zeta}_j = d\xi_j - i d\eta_j.$$

If the terms $\overline{\zeta}_j$ do not appear in this expansion, and if the coefficients of the expansion are holomorphic functions, we will say that the differential form is *holomorphic*: this notion is obviously invariant under analytic changes of charts, and could in any case have been defined intrinsically.

6 Partitions of unity on a $\mathscr{C}^\infty$ manifold

The notion of a "partition of unity" plays an essential role in "local to global" problems on manifolds.

6.1 Theorem. *For every open cover $\{U_i\}_{i\in I}$ of a $\mathscr{C}^\infty$ manifold X, where I is any set of indices, there exists a locally finite, $\mathscr{C}^\infty$ partition of unity which is subordinate to this cover. By this we mean a family of smooth $\mathscr{C}^\infty$ functions $\{\pi_i(x)\}_{i\in I}$ such that:*

(π 1) $\pi_i \geqslant 0, \quad \sum_i \pi_i = 1$;
(π 2) *Every point on X has a neighbourhood on which only finitely many of the functions π_i are supported;*
(π 3) *the support of π_i is contained in U_i.*

Remarks.

(i) Condition (π 2) ensures that the sum in (π 1) makes sense.
(ii) Condition (π 2) implies that the number of π_i which are not identically zero is finite or countable.

6.2 Corollary. *If A is a closed subset of X, and U is an open subset of A, then there exists a $\mathscr{C}^\infty$ function whose support is contained in U, whose values are between 0 and 1, and which is equal to 1 in a neighbourhood of A.*

It suffices to apply the theorem to the open cover of X which consists of the open sets U and $X - A$.

Sketch of proof of the theorem. We show, for any manifold which satisfies property (X_1) (§1), that every open cover $\{U_i\}$ has a locally finite "refinement" (3): in other words, there exists a *locally finite* open cover $\{U'_j\}$ such that every U'_j is contained in at least one U_i. We can even ensure that every $\overline{U'_j}$ is compact and contained in the domain of a local chart on X. In the same way, we can refine this open cover to obtain a cover $\{U''_j\}$ such that $\overline{U''_j} \subset U'_j$, and for each j we construct a $\mathscr{C}^\infty$ function $\varphi_j \geqslant 0$, supported on U'_j, and equal to 1 on $\overline{U''_j}$ (since we are in the domain of a local chart, it suffices to construct such a function on domains in a Euclidean space). Since the sets U''_j form a cover of X, the sum $\varphi = \sum_j \varphi_j$ is everywhere $\geqslant 1$, and therefore > 0, and the functions $\pi_j = \varphi_j/\varphi$ form a locally finite partition of unity subordinate to the cover $\{U'_j\}$. It is trivial to deduce the existence of a partition of unity subordinate to the cover $\{U_i\}$, simply by collecting terms.

6.3 An application: construction of a Riemannian metric

A $\mathscr{C}^\infty$ Riemannian metric on a manifold X is given by specifying a *symmetric positive-definite tensor* $G(x)$ at each point $x \in X$ (that is, a symmetric positive-definite bilinear function on pairs of tangent vectors), *which varies as a* $\mathscr{C}^\infty$ *function of* x. Let us show that such a metric always exists on a $\mathscr{C}^\infty$ manifold: if $\{h_i : U_i \to E_i\}$ is an atlas on X, such a tensor $G_i(x)$ can be constructed on each domain U_i (for example, one can take the tensor represented by the identity matrix in the coordinates of the chart h_i); we then choose a $\mathscr{C}^\infty$ partition of unity $\{\pi_i\}$ which is subordinate to the cover $\{U_i\}$, and we set

$$G(x) = \sum_i \pi_i(x) G_i(x).$$

6.4 An application of 6.3: construction of a tubular neighbourhood of a closed submanifold

Let $S \subset X$ be a closed $\mathscr{C}^\infty$ submanifold. Choose a $\mathscr{C}^\infty$ Riemannian metric on X, and to every point $y \in S$ associate a family of *geodesic arcs starting at* y*, of length* $\varepsilon(y)$*, and orthogonal to* S (where ε is a continuous function of y). As y ranges over S, this family sweeps out a neighbourhood V of S, and if the "radius" $\varepsilon(y)$ of this neighbourhood is taken to be sufficiently small,[4] then there will be a unique arc in the family which passes through each point $x \in V - S$ (this is true, but will be assumed here). If we let $\mu(x) \in S$ be the "base" of this arc, we can in this way define a map $\mu : V - S \to S$, which can be extended ($\mathscr{C}^\infty$ smoothly) by the identity map on S. We say that V is a "*tubular neighbourhood*" of the submanifold S, and that $\mu : V \to S$ is a $\mathscr{C}^\infty$ "*retraction*" of this neighbourhood onto S.

A useful corollary of the existence of such a retraction map is the fact that *every differential form* $\varphi \in \Omega(S)$ *is the restriction of a differential form* $\psi \in \Omega(X)$. Let π be a $\mathscr{C}^\infty$ function supported on V, and equal to 1 on S (cf. corollary 6.2). The differential form $\pi \cdot \mu^*\varphi$ is supported on V and can therefore be extended by zero $\mathscr{C}^\infty$ smoothly along the exterior of V, so let $\psi \in \Omega(X)$ be the differential form obtained in this way. This form obviously answers the question, because

$$\psi|S = \pi \cdot \mu^*\varphi|S = (\pi|S) \cdot \mu^*\varphi|S = \mu^*\varphi|S = \varphi \quad \text{since} \quad \mu|S = \mathbb{1}_S$$

(the identity $\mu^*\varphi|S = (\mu|S)^*\varphi$ can be immediately deduced from the functorial properties in §5.3).

[4] If S is compact, we can choose a radius ε which is independent of y.

7 Orientation of manifolds. Integration on manifolds

7.1 Orientation of a manifold

To "*orient*" a manifold X is to associate a number $\varepsilon_h = \pm 1$ to every local chart $h : U \to E$ of X, in such a way that for every pair of such charts

$$h : U \longrightarrow E, \quad h' : U' \longrightarrow E',$$

the sign of the determinant of the Jacobian $\operatorname{Det} T_\xi(h' \circ h^{-1})$ in $h(U \cap U')$ is equal to $\varepsilon_h \varepsilon_{h'}$. If there exists an orientation of X, it is clear that one can construct an atlas of X in which all the charts can be "patched together" using transformations whose Jacobians are positive. Such an atlas will be called an "*oriented atlas*". Conversely, *every oriented atlas* $\{h_i : U_i \to E_i\}$ *determines an orientation*: let $h : U \to E$ be any local chart, and let $\{h'_j : U'_j \to E'_j\}$ be the atlas of U defined by restricting the given atlas (the sets U'_j are the connected components of $U_i \cap U$). The open sets U'_j fall into two classes, those for which the Jacobian $\operatorname{Det} T_\xi(h^{-1} \circ h'_j)$ is negative, and those for which it is positive, and one easily checks (because the atlas $\{h'_j\}$ is oriented) that the open sets in the first class do not meet those in the second. Therefore, since U is connected, one of these two classes is empty. If it is the first, we set $\varepsilon_h = 1$, and if it is the second, we set $\varepsilon_h = -1$.

Let ε and ε' be two orientations on X. The local charts on X split into two classes, according to whether $\varepsilon_h = -\varepsilon'_h$ or $+\varepsilon'_h$, and as previously, we show that the domains of these two classes of charts do not intersect. Therefore *if X is connected*, one of these two classes is empty: if it is the first, $\varepsilon = \varepsilon'$; if it is the second, $\varepsilon = -\varepsilon'$. We have therefore shown that *if a connected manifold has an orientation ε, then it has exactly two orientations*, namely ε and $-\varepsilon$.

7.2 Example of a non-orientable manifold: real projective space P^n, if n is even

Consider the atlas $\{h_i : U_i \to \mathbb{R}^n, i = 0, 1, \ldots, n\}$ of P^n defined in §2.4 (ii).

An orientation ε of P^n determines a set of numbers ε_{h_i}. But the Jacobian of the transformation $h_1 \circ h_0$ is equal to $1/\xi_1^{n+1}$. Therefore, if n is even, the sign of this determinant changes between the two connected components $\{\xi_1 > 0\}$ and $\{\xi_1 < 0\}$ of $U_0 \cap U_1$, and therefore cannot be equal to the sign of $\varepsilon_{h_0} \varepsilon_{h_1}$ over the whole of $U_0 \cap U_1$.

7.3 Every complex analytic manifold has a canonical orientation

This follows since every analytic map $f : \mathbb{C}^n \to \mathbb{C}^n$ gives rise to a map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ whose Jacobian determinant is *positive*: a quick way to see this is to observe that the real Jacobian $T_{(\xi,n)}(f)$ can be formally written

$$\begin{array}{|c|c|}\hline \dfrac{\partial f_i}{\partial \zeta_j} & 0 \\ \hline 0 & \dfrac{\partial \overline{f}_i}{\partial \overline{\zeta}_j} \\ \hline \end{array}$$

in the variables

$$\zeta = \xi + i\eta, \quad \overline{\zeta} = \xi - i\eta;$$

with

$$\frac{\partial \overline{f}_i}{\partial \overline{\zeta}_j} = \left(\overline{\frac{\partial f_i}{\partial \zeta_j}}\right).$$

As a result, *every complex analytic atlas on a manifold X is oriented*, and this canonically determines the orientation of the manifold.

7.4 Definition of an orientation using n tangent vectors

If ε is an orientation of an n-dimensional manifold X, we can assign to every system of n *linearly independent* tangent vectors $V_1, V_2, \ldots, V_n \in T_x(X)$, the number

$$\varepsilon(V_1, V_2, \ldots, V_n) = \varepsilon_h \operatorname{sgn} \operatorname{Det} \|v_{ij}\| = \pm 1,$$

where $\|v_{ij}\|$ is the $n \times n$ matrix given by the coordinates of the vectors $V_1, V_2, \ldots, V_n$ in a chart h defined on a connected neighbourhood of x.

It is obvious that this number does not depend on the chart h and that *if the variety X is orientable*, its orientation is completely determined by specifying *one* point x and *one* system $V_1, V_2, \ldots, V_n \approx T_x(X)$ which satisfies $\varepsilon(V_1, V_2, \ldots, V_n) = +1$. Such a system will be called a "*system of indicators for the orientation of X at the point x*".

7.5 Definition of an orientation using a differential form of degree n

Suppose that we are given a differential form φ of degree n on an n-dimensional manifold X *which vanishes nowhere*. If $\xi_1, \xi_2, \ldots, \xi_n$ are local coordinates in a chart $h : U \to E$, then on U, φ can be written:

$$\varphi = \rho_h(x) d\xi_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_n.$$

The function $\rho_h(x)$ does not vanish on the connected set U, and thus has a constant sign. If ε_h denotes its sign, then one verifies immediately that the correspondence $h \rightsquigarrow \varepsilon_h$ is an *orientation* on X.

Conversely, suppose that we are given an orientation ε on X. Having chosen a Riemannian metric $G(x)$ (§6.3) on X, we can assign to every system of n tangent vectors $V_1, V_2, \ldots, V_n \in T_x(X)$ the "volume" (with respect to the metric G) of the parallelopiped formed by these n vectors,[5] multiplied by the sign $\varepsilon(V_1, V_2, \ldots, V_n)$. In this way we have defined a differential form of degree n which vanishes nowhere: it will be called the "fundamental form" of the oriented manifold X with respect to the metric G.

7.6 The integral of a differential form on an oriented manifold

Let X be an oriented manifold of dimension n, and let φ be a differential form on X of degree n with compact support.

We define the *integral of* φ *on* X, denoted $\int_X \varphi$ (strictly speaking, we should really write $\int_{X,\varepsilon} \varphi$, where ε denotes the orientation of X).

Suppose to begin with that the support of φ is contained in the domain of definition U of a local chart h. If $\xi_1, \xi_2, \ldots, \xi_n$ denote the coordinates of this chart, φ can be written

$$\varphi = \rho_h(\xi) d\xi_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_n.$$

We therefore set

$$\int_X \varphi = \varepsilon_h \iint \ldots \int d\xi_1 d\xi_2 \ldots d\xi_n \, \rho_h(\xi).$$

This integral obviously converges because ρ_h has compact support. Furthermore, it is independent of the choice of chart h. To see this, observe that a change of charts $h \rightsquigarrow h'$ multiplies the form ϕ by the Jacobian of the map $h \circ h'^{-1}$, in such a way that $\varepsilon_h \rho_h$ is multiplied by the *absolute value* of this Jacobian, and this precisely agrees with the change of variables rule for integrals.

Let us now do the general case. The support of φ (which is *compact*) can be covered by a *finite* number of domains U_i equipped with charts h_i. Let $\{\pi_i\}$ be a partition of unity which is subordinate to this covering. We set

$$\int_X \varphi = \sum_i \int_X \pi_i \varphi$$

and it is easy to verify that this expression does not depend on the choice of the partition of unity.

Remark. More generally, if the support of φ is not compact, we say that the integral $\int_X \varphi$ "*converges*" if the (infinite) series $\sum_i \int_X \pi_i \varphi$ is *summable* for every locally finite partition of unity $\{\pi_i\}$ with compact supports. In that case we define $\int_X \varphi$ to be the value of this series (this value is obviously independent of the choice of partition, because of the summability assumption).

[5] This volume is defined by the expression $(\mathrm{Det}\, \|G\,(V_i, V_j\|_{i,j})^{1/2}$.

7.7 Integration on a manifold with boundary and Stokes' formula

In order to define a "*manifold with boundary*" $\overline{X}$, it suffices to repeat the definition of an ordinary manifold word for word, replacing Euclidean space $\mathbb{R}^n$ by "closed Euclidean half-space"

$$\overline{\mathbb{R}^n_-} = \{x_1, x_2, \dots, x_n : x_1 \leqslant 0\}\cdot$$

Given a "local chart" on $\overline{X}$, the points of $\overline{X}$ separate into two classes: those which are mapped onto

$$\mathbb{R}^n_- = \{x_1, x_2, \dots, x_n : x_1 < 0\}$$

and those which are mapped onto the hyperplane

$$\partial\overline{\mathbb{R}^n_-} = \{0, x_2, \dots, x_n\}\,.$$

One can show that this partition of $\overline{X}$ is independent of the local chart under consideration (this is an immediate consequence of the theorem on the *topological invariance of open sets* (4)). We therefore obtain a partition of X into two *manifolds* X and $\partial\overline{X}$, of dimension n and $n-1$ respectively, known as the *interior* and *boundary* of X (these are manifolds in the ordinary sense, i.e., without boundary).

A *differentiable structure* on a manifold with boundary is defined as for an ordinary manifold. Let us nonetheless make it clear that a "differentiable map" on $\overline{\mathbb{R}^n_-}$ should mean a map which can be extended to a differentiable map on $\mathbb{R}^n$. In this way, the differentiable structure on $\overline{X}$ induces differentiable structures on X and $\partial\overline{X}$ by restriction. In the same way, an orientation of $\overline{X}$ induces orientations on X and $\partial\overline{X}$ (in the case of $\partial\overline{X}$, we have canonically identified $\partial\overline{\mathbb{R}^n_-}$ with $\mathbb{R}^{n-1}$: $\{0, x_2, \dots, x_n\} \rightsquigarrow \{x_2, \dots, x_n\}$).

We define in the obvious way the *tangent space* of $\overline{X}$ (it is a vector space of dimension n), the set of *differential forms* on $\overline{X}$, and, if $\overline{X}$ is oriented, the integral of a differential form with compact support.

Naturally,

$$\int_{\overline{X}} \varphi = \int_X \varphi | X.$$

Stokes' formula. *If φ is a differential form on X with compact support,*

$$\boxed{\int_{\partial\overline{X}} \varphi|\partial\overline{X} = \int_{\overline{X}} d\varphi.}$$

Proof. Using local charts and a partition of unity, we can reduce to the case where $\overline{X} = \mathbb{R}^n_-$. The form φ (assumed to be of degree $n-1$) can therefore be written

$$\varphi(x) = \sum_{i=1}^{n} \rho_i(x) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

and therefore

$$d\varphi(x) = \left(\sum_{i=1}^{n} (-)^{i-1} \frac{\partial \rho_i(x)}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n,$$

which can immediately be integrated. □

8 Appendix on complex analytic sets

We refer to [1] and [15] for the results mentioned in this section.

8.1

A subset S of a complex analytic manifold X is said to be *analytic at* a if there exists a neighbourhood U_a of a and a finite number of analytic functions $s_1, s_2, \ldots, s_k : U_a \to \mathbb{C}$ called "*local equations* of S", such that

$$S \cap U_a = \{x \in U_a : s_1(x) = \cdots = s_k(x) = 0\}\cdot$$

The subset S is called *analytic* if it is analytic at every point of X. It is then a *closed subset* of X.

The *codimension* of an analytic set S at the point a is denoted $\operatorname{codim}_a S$ and is the largest integer q such that one can find a complex analytic submanifold of dimension q which intersects S at the point a and *at the point* a *only*. The number k of local equations required to define S is clearly at least equal to the codimension of S, but in general it will be greater except for very simple sets S. The *dimension* of S at a is $\dim_a S = n - \operatorname{codim}_a S$, where n denotes the dimension of X. We will assume without proof the fact that if S is contained in a submanifold $Y \subset X$, the dimension of S considered as a subset of Y is equal to its dimension as a subset of X. A further interesting property is that for x sufficiently near a on S, $\dim_x S \leqslant \dim_a S$.

8.2 Analytic sets defined by an ideal

Since all the following notions are local, we will assume that $X = \mathbb{C}^n$.

Let $\mathscr{O}_a$ be the ring of *germs of analytic functions at* a (i.e., Taylor series at the point a in n variables with non-zero radius of convergence). Let $\mathfrak{S}_a \subset \mathscr{O}_a$ be an *ideal* in $\mathscr{O}_a$ (i.e., a family of elements which is stable under addition and such that $\forall f_a \in \mathscr{O}_a$, $f_a \cdot \mathfrak{S}_a \subset \mathfrak{S}_a$. One can show that the ring $\mathscr{O}_a$ is *Noetherian*, in other words, all its ideals have a *finite* family of generators.

Now, every finite system of generators of the ideal $\mathfrak{S}_a$ can be used to define a system of local equations for an analytic set, defined in the common domain of convergence U_a of these generators. On the other hand, enlarging the system of generators (i.e., adding a finite number of elements of the ideal to the initial system) may make the common domain of convergence smaller, but in no way modifies the analytic set in this smaller domain. The ideal $\mathfrak{S}_a$ thus corresponds to the *germ of an analytic set at* a (i.e., an analytic set defined in an *unspecified* neighbourhood of a), denoted $S(\mathfrak{S}_a)$.

8.3 The ideal of an analytic set

If S is a set which is analytic at a, the set of all germs of analytic functions which vanish on S is clearly an *ideal* in $\mathscr{O}_a$, denoted $i_a(S)$. If S was given by some ideal $\mathfrak{S}_a$, how can one deduce the ideal $i_a(S(\mathfrak{S}_a))$ from it? The answer is given by the

Nullstellensatz (Hilbert–Rückert). $i_a(S(\mathfrak{S}_a))$ is the "radical" of $\mathfrak{S}_a$, i.e., the set of germs of functions which belong to $\mathfrak{S}_a$ after being raised to a certain power.

8.4 The rank of an ideal

We define the rank of an ideal $\mathfrak{S}_a \subset \mathscr{O}_a$ to be the maximum rank of every Jacobian matrix which can be formed out of functions whose germs belong to this ideal:

$$\operatorname{rank} \mathfrak{S}_a = \sup_{s_{1a},\ldots,s_{ma} \in \mathscr{S}_a} \operatorname{rank} \frac{D(s_1,\ldots,s_m)}{D(\xi_1,\ldots,\xi_n)}(a)$$

(this number is obviously finite because the ring $\mathscr{O}_a$ is Noetherian).

Proposition. *$S(\mathfrak{S}_a)$ is contained in a submanifold of codimension equal to the rank of $\mathfrak{S}_a$.*

Proof. By definition of the rank r of $\mathfrak{S}_a$, we can extract r functions $s_1, s_2, \ldots, s_r$ from the ideal $\mathfrak{S}_a$ whose Jacobian is of rank r at a (and therefore in a neighbourhood of a). By §4.5, these r functions are the local equations of a submanifold of codimension r, which obviously contains $S(\mathfrak{S}_a)$. □

Corollary. Rank $\mathfrak{S}_a \leqslant \operatorname{codim} S(\mathfrak{S}_a)$.

In the special case when *equality holds*, i.e., $\operatorname{rank} \mathfrak{S}_a = \operatorname{codim} S(\mathfrak{S}_a) = r$, we will show that $S(\mathfrak{S}_a)$ is a germ of a *submanifold of codimension* r. In fact, by using the proposition above, we can reduce to the case where $r = 0$. We must therefore show that an analytic subset of codimension zero in $\mathbb{C}^n$ is equal to the whole of $\mathbb{C}^n$, which follows immediately from the definition of codimension: since analytic functions in one complex variable can only have isolated zeros (unless they are identically zero), every line must cut an analytic set in isolated points, unless it is completely contained in it.

8.5 Smooth points of an analytic set

Definitions. A point $a \in S$ is called *smooth* if $\operatorname{rank} i_a(S) = \operatorname{codim}_a S$; otherwise, it is called *singular point*.

Proposition. *$a \in S$ is smooth $\Leftrightarrow$ S is a submanifold near a.*

Proof. ($\Rightarrow$) was already proved in the previous section [here, we take $\mathfrak{S}_a$ to be the whole ideal $i_a(S)$].

($\Leftarrow$): If S is an analytic submanifold of codimension r, we can choose coordinates $z_1, z_2, \dots, z_r$ near a, such that $S = \{z : z_1 = \cdots = z_r = 0\}$. The ideal spanned by $z_1, z_2, \dots, z_r$ at the point a (the origin of the coordinates) is of rank r, and is obviously equal to its own radical: it is therefore $i_a(S)$. $\square$

8.6 Irreducibility

The following definitions can just as well be applied *locally* (to germs of analytic sets) as *globally* (to analytic sets). An analytic set S is said to be *irreducible* if it cannot be written as the union of two analytic sets which are distinct from it. Otherwise, it is said to be *reducible*. An *irreducible component* of S is a maximal element in the family of irreducible analytic sets contained in S.

II

Homology and cohomology of manifolds

1 Chains on a manifold (following de Rham). Stokes' formula

1.1

A p-dimensional "*chain element*" $[\sigma]$ on a differentiable manifold X, is defined by a *convex polyhedron* Δ^p of dimension p, an *orientation* ε of Δ^p, and a *differentiable map* $\sigma : \Delta^p \to X$.

More precisely, Δ^p is a compact convex polyhedron in a p-dimensional affine space E^p, and σ is the restriction of a differentiable map defined in a neighbourhood of this compact set.

If φ is a differential form of degree p on X, the *integral of* φ *along the chain element* $[\sigma]$ is defined by

$$\int_{[\sigma]} \varphi = \int_{\Delta^p} \sigma^* \varphi.$$

The integral over the oriented polyhedron Δ^p obviously makes sense, because $\sigma^*\varphi$ is the restriction to the compact set Δ^p of a differential form which is $\mathscr{C}^\infty$ in a neighbourhood of this compact set.

1.2

A *p-dimensional chain* γ in X is defined by a finite formal linear combination of p-dimensional chain elements with integer coefficients. Moreover, we say that two such linear combinations

$$\sum_i n_i[\sigma_i] \quad \text{and} \quad \sum_j n'_j[\sigma'_j]$$

define *the same chain* γ if, for any form φ, they give rise to the same integral:

$$\int_\gamma \varphi = \sum_i n_i \int_{[\sigma_i]} \varphi = \sum_j n'_j \int_{[\sigma'_j]} \varphi.$$

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_2,

1.3 The boundary

Let $[\sigma] = (\Delta^p, \varepsilon, \sigma)$ be a p-dimensional chain element. Every face Δ^p of the polyhedron Δ^p is contained in a $(p-1)$-plane E_i^p, which we can equip with an orientation ε_i, defined as follows:

$$\varepsilon_i\left(V_1, V_2, \ldots, V_{p-1}\right) = \varepsilon\left(V_0, V_1, V_2, \ldots, V_{p-1}\right),$$

where V_0 is the exterior normal vector to the face E_i^p.

If we set $\sigma_i = \sigma|\Delta_i^p$, we obtain in this way a chain element

$$[\sigma_i] = (\Delta_i^p, \varepsilon_i, \sigma_i).$$

The sum of all the chain elements which correspond to all the faces of Δ^p is called the *boundary* of $[\sigma]$, and is denoted $\partial[\sigma]$. For any *chain* $\gamma = \sum_i n_i[\sigma_i]$, we set $\partial\gamma = \sum_i n_i\partial[\sigma_i]$. It will follow immediately from Stokes' formula (§1.4) that this chain $\partial\gamma$ does not depend on the particular representation $\sum_i n_i[\sigma_i]$ of γ, and furthermore, that $\partial\partial\gamma = 0$ (which follows easily from $dd = 0$).

1.4 Stokes' formula.

$$\boxed{\int_{\partial\gamma} \varphi = \int_\gamma d\varphi.}$$

It suffices to prove this formula when γ is a chain element $[\sigma]$. The general case follows by linearity. Setting $\psi = \sigma^*\varphi$, we are reduced to the formula

$$\int_{\Delta^p} d\psi = \sum_i \int_{\Delta_i^p} \psi,$$

which can be verified in the same way as the formula in §I.7.7 (it is in fact the same formula, except that Δ^p is a manifold whose boundary is *piecewise* differentiable, whereas in §I.7.7 we assumed that the boundary was differentiable).

1.5 Chain transformations

Let $f : X \to Y$ be a differentiable map of manifolds. To every chain element $[\sigma] = (\Delta^p, \varepsilon, \sigma)$ in X, we will associate the chain element $f_*[\sigma] = (\Delta^p, \varepsilon, f \circ \sigma)$ on Y, and likewise, to every chain $\gamma = \sum_i n_i[\sigma_i]$ in X, we associate the chain $f_*\gamma = \sum_i n_i f_*[\sigma_i]$ in Y.

Obviously, $\int_{f_*\gamma} \varphi = \int_\gamma f^*\varphi$ (which is an immediate consequence of §I.5.3, which says that $(f \circ \sigma)^*\varphi = \sigma^* f^*\varphi$).

This identity proves that the chain $f_*\gamma$ is indeed independent of the choice of representative for γ. Moreover, combined with Stokes' formula and the property $df^* = f^*d$ of §I.5.3, it proves that

$$\boxed{\partial f_* = f_*\partial.}$$

This follows from:

$$\int_{\partial f_*\gamma} = \int_{f_*\gamma} d\varphi = \int_\gamma f^* d\varphi = \int_\gamma df^*\varphi = \int_{\partial\gamma} f^*\varphi = \int_{f^*\partial\gamma} \varphi.$$

1.6 Example. Let $X = S^2$ denote the unit sphere in Euclidean space $\mathbb{R}^3$. The parametric representation of this sphere in polar coordinates defines a two-dimensional chain element $[\sigma] = (\Delta, \varepsilon, \sigma)$ (Δ is a rectangle of length 2π and width π, and $\sigma : \Delta \to S^2$ is the map defined by "longitudinal–latitudinal" coordinates). The integral of any form φ on this chain element is therefore equal to the integral of φ on the sphere S^2, equipped with the orientation ε_{S^2} "corresponding" to ε ($\varepsilon_{S^2} \circ \sigma = \varepsilon$). In particular, let φ be the differential form corresponding to the "solid angle", in other words, the fundamental form (cf. §I.7.5) associated to the orientation ε_{S^2} and to the metric on S^2 which is induced by the Euclidean metric on $\mathbb{R}^3$. We have

$$\int_{[\sigma]} \varphi = 4\pi.$$

Observe that $d\varphi = 0$ (since φ is of maximal degree) and that $\partial[\sigma] = 0$. In this case we say that φ is a *closed form* and that $[\sigma]$ is a *cycle*. However, φ is not an "exact differential form", nor is $[\sigma]$ a "boundary". This means that there does not exist a form ψ such that $\varphi = d\psi$, nor does there exist a chain γ such that $[\sigma] = \partial\gamma$. If either were to exist, then Stokes' formula would say that $\int_{[\sigma]} \varphi = 0$.

2 Homology

2.1

The main point of the concept of a "chain" is to act as a stepping stone towards the construction of "homology groups", which we are about to define. These days it is preferable to use a slightly different definition of chains from the one given in the previous section, which has the added advantage of being valid for any topological space. On a manifold, all these definitions give rise to the same homology groups, but the modern definition enables us to extend what was previously only possible for differentiable maps, to the case of *continuous maps.*

Let us therefore state some general properties for chains which were obtained in the previous section for differentiable maps between manifolds.

To every topological space X is associated an *abelian group* $C_*(X)$, called the *group of chains* of X. This group is graded by the "dimension" p of the chains:

$$C_*(X) = \bigoplus_p C_p(X)$$

and is equipped with a "*boundary homomorphism*":

$$\partial : C_*(X) \longrightarrow C_*(X)$$

which decreases the dimension by one, and satisfies

$$\partial\partial = 0.$$

To every continuous map $f : X \to Y$ one associates a homomorphism $f_* : C_*(X) \to C_*(Y)$ which preserves the dimension of chains, and which commutes with ∂. The map $f \rightsquigarrow f_*$, is a *covariant functor* (cf. §I.4.2).

2.2 Definition (of homology). Let $Z_*(X) = \operatorname{Ker} \partial$ denote the "*kernel*" of the homomorphism ∂, i.e., the set of chains $\gamma \in C_*(X)$ such that $\partial\gamma = 0$. Such a chain is called a *cycle*, and $Z_*(X)$ is the *group of cycles*.

Let $B_*(X) = \operatorname{Im} \partial$ denote the *image* of the homomorphism ∂, i.e., the set of chains of the form $\partial\gamma$. Such a chain is called a *boundary*, and $B_*(X)$ is the *group of boundaries*.

Since $\partial\partial = 0$, $B_*(X)$ is subgroup of $Z_*(X)$. We can therefore consider the quotient

$$H_*(X) = Z_*(X)/B_*(X)$$

which is called the *homology group* of the space X. Its elements are called "homology classes". Two cycles belong to the same homology class if the difference between them is a boundary, and in this case we say that the cycles are "homologous".

All the above groups are clearly graded by the dimension of the chains which define them. Thus,

$$H_*(X) = \bigoplus_{p=0,1,2,\ldots} H_p(X).$$

In dimension zero, we set $Z_0(X) = C_0(X)$, so that $H_0(X) = C_0(X)/B_0(X)$. But $C_0(X)$ is the *free group generated by the set* X (the group of formal linear combinations of points of X), and $B_0(X)$ is spanned by linear combinations of the form $[x] - [x']$ in $C_0(X)$, where x and x' are the endpoints of a *path* on X. Therefore $H_0(X)$ is the free group generated by the set of *path-connected components of the space* X.

2.3 Examples.

(i) Let $X = S^n$ be the unit sphere of dimension n in $\mathbb{R}^{n+1}$. It is connected, so $H_0(S^n) \approx \mathbb{Z}$ (the group of integers $\gtrless 0$). Otherwise, one shows that $H_*(S^n)$ is zero in all other dimensions, *except in dimension* n, where $H_n(S^n) \approx \mathbb{Z}$. This group is spanned by a cycle which can be constructed in a similar manner to the cycle $[\sigma]$ in example 1.6.

(ii) Let $X = P^n$ be real n-dimensional projective space. Obviously, $H_0(P^n) \approx \mathbb{Z}$. Otherwise, one can prove that $H_{2p}(P^n) = 0$ and that $H_{2p+1}(P^n) \approx \mathbb{Z}_2$ (the cyclic group of order 2), except in the case $2p+1 = n$, where $H_{2p+1}(P^{2p+1}) \approx \mathbb{Z}$. Intuitively, the generator in dimension $2p+1$ "corresponds" to a $(2p+1)$-plane $P^{2p+1} \subset P^n$.[1]
One can also construct chains corresponding to $2p$-planes $P^{2p} \subset P^n$, but, because of the non-orientability of P^{2p} (§7.2), one notices that these chains are not cycles but have a boundary corresponding to a copy of P^{2p-1} which is covered twice. It is this fact which explains why the homology groups in dimension $2p-1$ are cyclic groups of order 2 (P^{2p-1} is not a boundary, but 2 P^{2p-1} is).

2.4 Torsion

In §2.3 (ii) above we gave an example of a homology group with "*torsion*". We say that an abelian group G has torsion if it contains cyclic elements of finite order.

In most cases which arise in practice, homology groups have a *finite* number of generators, and are therefore[2] isomorphic to a direct sum of cyclic groups of finite order ($\mathbb{Z}_r$), or infinite order ($\mathbb{Z}$). If we discard the groups of finite order, we obtain a *free group* (with a finite number of generators), which is called the "homology group *modulo torsion*". The case where there are an infinite number of generators is a little delicate,[3] and we will (implicitly) put this question aside whenever we speak about torsion.

2.5

Let $f : X \to Y$ be a continuous map. Since the homomorphism $f_* : C_*(X) \to C_*(Y)$ of §2.1 commutes with ∂, it maps cycles to cycles, and boundaries to boundaries. It therefore induces a homomorphism $f_* : H_*(X) \to H_*(Y)$ between homology groups which preserves dimensions. The map $f \rightsquigarrow f_*$ is a *covariant functor*. From this it follows in particular that if $f : X \approx Y$ is a homeomorphism, $f_* : H_*(X) \approx H_*(Y)$ is an isomorphism of abelian groups.

2.6 Retractions

A continuous map $r : X \to A$ is called a *retraction* if:

1° A is a subspace of X,
2° $r|A = \mathbb{1}_A$.

[1] In other words, it can be represented by a cycle constructed from a parametric representation of a $(2p+1)$-plane.
[2] This is an exercise in algebra, which is not so straightforward.
[3] For example, a torsion-free group is not necessarily free.

Let $i : A \to X$ be the "inclusion" from A into X. Condition 2° can be written $r \circ i = \mathbb{1}_A$, from which it follows by functoriality that $r_* i_* = \mathbb{1}_{H_*(A)}$

$$\begin{tikzcd} & H_*(X) \arrow[dr, "r_*"] & \\ H_*(A) \arrow[ur, "i_*"] \arrow[rr, "\mathbb{1}"] & & H_*(A) \end{tikzcd}$$

This diagram shows that $H_*(A)$ is a *direct factor* of the group $H_*(X)$ [in other words $\exists\, G \ : \ H_*(X) \approx H_*(A) \times G$].

2.7 Example (Brouwer's fixed point theorem). Let E^n denote the closed unit ball in $\mathbb{R}^n$. Its boundary is a sphere S^{n-1}. The space E^n is "homologically trivial" (§2.9), which is not the case for S^{n-1} [§2.3 (i)]. Therefore, $H_*(S^{n-1})$ cannot be a subgroup of $H_*(E^n)$, so *there cannot exist a retraction from E^n onto S^{n-1}*.

Brouwer's "fixed point theorem" states that *every continuous map $f : E^n \to E^n$ has at least one fixed point*, which is a corollary of the previous statement. To see this, suppose that the contrary is true, i.e., $\forall\, x \in E^n$, $f(x) \neq x$, and let $r(x)$ denote the point where the sphere S^{n-1} meets the half-line $\overline{f(x), x}$ (Fig. II.1). It is easy to check that the map $r : E^n \to S^{n-1}$ which is defined in this way is continuous, and clearly $r|S^{n-1} = \mathbb{1}_{S^{n-1}}$. This would mean that r is a retraction, contradicting the statement above.

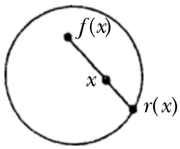

Fig. II.1.

2.8 Homotopy

Two continuous maps f_0 and $f_1 : X \to Y$ are called *homotopic* (in short, $f_0 \simeq f_1$) if they can be "interpolated" by a continuous map $f_\tau : X \to Y$.[4]

One can prove the following fundamental property:

Two homotopic maps give rise to the same homomorphism between homology groups:

$$f_0 \simeq f_1 \Longrightarrow \boxed{f_{0_*} = f_{1_*}} : H_*(X) \longrightarrow H_*(Y).$$

[4] We say that the family of continuous maps $f_\tau : X \to Y$ ($\tau \in [0,1]$) "interpolates" f_0 and f_1, if $f(x,\tau) = f_\tau(x)$ defines a continuous map $f : X \times [0,1] \to Y$.

2.9 Deformation-retractions

A retraction $r : X \to A$ (cf. §2.6) is called a *deformation-retraction* if $i \circ r \simeq \mathbb{1}_X$. It follows from functoriality, together with homotopy invariance 2.8, that $i_* r_* = \mathbb{1}_{H_*(X)}$. By 2.6, we deduce that $r_* : H_*(X) \to H_*(A)$ is an *isomorphism* (and i_* is the inverse isomorphism).

Important special case. The space X is said to be "*contractible*" if there is a deformation retraction onto one of its points. It is therefore "*homologically trivial*", i.e., it has the homology of a point:

$$H_p(X) = 0 \quad \text{if } p > 0,$$
$$H_0(X) = \mathbb{Z}.$$

2.10 Examples. *Euclidean space $\mathbb{R}^n$ is contractible.* If $r : \mathbb{R}^n \to P$ is the retraction from $\mathbb{R}^n$ onto the origin P, the relation

$$f_\tau(x_1, x_2, \ldots, x_n) = (\tau x_1, \tau x_2, \ldots, \tau x_n)$$

defines a *homotopy* between

$$i \circ r = f_0 \quad \text{and} \quad \mathbb{1}_{\mathbb{R}^n} = f_1.$$

The same argument shows that *every star domain in $\mathbb{R}^n$ is contractible.* In the same way, it is easy to see that a cylindrical surface deformation retracts onto its base, and so on.

2.11 Relative homology

Let (X, A) be a *pair*, in other words, a topological space X and a subspace A. Let $i_* : C_*(A) \to C_*(X)$ be the homomorphism induced on chains by the inclusion map $i : A \to X$.[5] The quotient group

$$C_*(X, A) = C_*(X)/i_* C_*(A)$$

is called the group of *relative chains* of the pair (X, A). The homomorphism ∂ clearly induces a homomorphism

$$\partial : C_*(X, A) \longrightarrow C_*(X, A)$$

which satisfies the same properties as in §2.1, and as a result, one can define the *relative homology* group to be

$$H_*(X, A) = Z_*(X, A)/B_*(X, A),$$

[5] This homomorphism is *injective*, and so one can identify $C_*(A)$ with a subgroup of $C_*(X)$.

where

$$Z_*(X,\, A) = \operatorname{Ker} \partial, \qquad B_*(X,\, A) = \operatorname{Im} \partial.$$

Observe that a "relative cycle" [an element of $Z_*(X,\, A)$] is represented by a chain of X whose boundary is a chain in A. This boundary is a cycle in A, and one can easily check that its homology class in A only depends on the relative homology class [in $(X,\, A)$] of the original relative cycle. We have thus defined a homomorphism

$$\partial_* : H_p(X,\, A) \longrightarrow H_{p-1}(A).$$

The *functorial* properties and the *homotopy* property also exist for relative homology: it suffices to replace the word "map" by "map of pairs" in the statements of §§1.5 and 2.8 [a "map of pairs" $f\colon (X,\, A) \to (Y,\, B)$ consists of two pairs $(X,\, A)$, $(Y,\, B)$ and a map $f : X \to Y$ such that $f(A) \subset B$].

Moreover, the homomorphism ∂_* transforms "naturally": for every map of pairs

$$f : (X,\, A) \longrightarrow (X,\, B),$$

the diagram

$$\begin{array}{ccc} H_p(X,\, A) & \xrightarrow{f_*} & H_p(Y,\, B) \\ \downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \partial_*} \\ H_{p-1}(A) & \xrightarrow{(f|A)_*} & H_{p-1}(B) \end{array}$$

commutes (5).

2.12 Remark (on coefficients). Instead of defining chains as being linear combinations of chain elements *with integer coefficients*, we could obviously have taken linear combinations with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (for example). In this way, by taking coefficients in a *field*, we replace all the *groups* (groups of chains, homology groups, etc.) with *vector spaces*. This has the effect of "killing" the torsion. If the "ordinary" homology group (i.e., with coefficients in $\mathbb{Z}$) has a finite number of generators, it defines, *modulo torsion* (§2.4), a *free group with a finite number of generators*, and the homology with coefficients in a field is simply the *vector space spanned by the same generators*.

3 Cohomology

3.1 Cochains

On every topological space X, there is the notion of a "*cochain*" which is *dual to the notion of a chain*, and this will be used to construct the "cohomology" of X. If X is a differentiable manifold, and if we are interested in its cohomology with coefficients in $\mathbb{R}$ or $\mathbb{C}$, we can replace cochains by *differential forms* which are defined globally on the manifold. The duality is given by integration (this is *de Rham's theorem*).

We will state some general properties of cochains, indicating the corresponding terminology for differential forms in brackets.

To every topological space X is associated an abelian group $C^*(X)$ [or vector space $\Omega(X)$] called the *group of cochains* of X [the space of differential forms on X]. This group is graded by the "*degree*" p of the cochains:

$$C^*(X) = \bigoplus_p C^p(X) \qquad [\Omega(X) = \bigoplus_p \Omega^p(X)],$$

and it is equipped with a "*coboundary homomorphism*" δ (differential d) which increases the degree by one, and satisfies

$$\delta\delta = 0 \qquad [dd = 0].$$

For every continuous map $f : X \to Y$, there corresponds a homomorphism $f : C^*(Y) \to C^*(X)$ which preserves the degree of cochains and commutes with δ. The correspondence $f \rightsquigarrow f^*$ is a *contravariant functor*.

3.2 Definition (of cohomology). In complete analogy with §2.2 we have:

- Cocycles: $Z^*(X) = \operatorname{Ker}\delta$ [closed differential forms: $\Phi(X) = \operatorname{Ker} d$];
- Coboundaries: $B^*(X) = \operatorname{Im}\delta$ [exact differential forms: $d\Omega(X)$];
- Cohomology: $H^*(X) = Z^*(X)/B^*(X)$ [$= \Phi(X)/d\Omega(X)$].

All these groups are graded by the *degree* of cochains:

$$H^*(X) = \bigoplus_{p=0,1,2,\ldots} H^p(X)$$

[if X is a manifold of dimension n, the direct sum stops at n].

In dimension zero, we set $B^0(X) = 0$, so that

$$H^0(X) = Z^0(X) \qquad [= \Phi^0(X)].$$

But $\Phi^0(X)$ is the vector space of differential functions on X with zero differential. Any such function is constant on every connected component of the manifold X. In this way, a basis for the vector space $H^0(X)$ is given by the *set of connected components* of X.

3.3

Every continuous map $f : X \to Y$ induces a homomorphism $f^* : H^*(Y) \to H^*(X)$ between cohomology groups which preserves the degree.

The correspondence $f \rightsquigarrow f^*$ is a *contravariant functor*.

Corollaries.

- *If $f : X \approx Y$ is a homeomorphism, $f^* : H^*(X) \approx H^*(Y)$ is an isomorphism;*
- *if X retracts onto A, $H^*(A)$ is a direct summand of $H^*(X)$.*

3.4 Homotopy

Two homotopic maps induce the same homomorphism between cohomology groups:

$$f_0 \simeq f_1 \Longrightarrow f_0^* = f_1^*.$$

Corollary. *A contractible space is cohomologically trivial, i.e., it has the same cohomology as a point:*

$H^p(X) = 0$ *if* $p > 0$;
$H^0(X) = \mathbb{C}$ *or* $\mathbb{R}$ *for cohomology with coefficients in* $\mathbb{C}$ *or* $\mathbb{R}$.

In particular, the fact that Euclidean space is cohomologically trivial goes by the name of the "*converse of Poincaré's theorem*": every closed differential form on Euclidean space is an exact differential.

3.5 Relative cohomology

Let (X, A) be a pair, and let us consider a *restriction homomorphism*[6] of cochains:

$$i^* : C^*(X) \longrightarrow C^*(A).$$

The *kernel* of this homomorphism

$$C^*(X, A) = \operatorname{Ker} i^*,$$

is called the group of *relative cochains* of the pair (X, A) [in the case of manifolds, this kernel will be denoted $\Omega(X, A)$: it is the vector space of *differential forms of* X *whose restriction to the submanifold* A *is zero*].

This group is obviously equipped with a "coboundary" homomorphism which satisfies the same properties as in §3.1, so that we can define its cohomology:

$$\begin{aligned} H^*(X, A) &= Z^*(X, A)/B^*(X, A) \quad [= \Phi(X, A)/d\Omega(X, A)], \\ Z^*(X, A) &= \operatorname{Ker} \delta, \qquad B^*(X, A) = \operatorname{Im} \delta. \end{aligned}$$

This cohomology is itself equipped with a "*coboundary homomorphism*"

$$\delta^* : H^p(A) \longrightarrow H^{p+1}(X, A)$$

defined as follows: let φ be a cocycle of A, which belongs to the cohomology class $h^p \in H^p(A)$, and let ψ be a cochain of X such that $i^*\psi = \varphi$ (if A is a closed differentiable submanifold of X, such a differential form ψ always exists by §I.6.4; in the general case, one shows that one can choose φ in h^p in such

[6] Recall (§I.6.4) that if A is a *closed* differentiable submanifold of X, the *restriction* homomorphism of differential forms is *surjective*.

a way that ψ exists). Then $\delta\psi$ defines a cocycle on X, whose restriction to A is zero:

$$\delta\psi|A = i^*\delta\psi = \delta i^*\psi = \delta\varphi = 0,$$

and is therefore a relative cocycle on (X, A). One checks that its relative cohomology class, denoted $\delta^* h^p \in H^{p+1}(X, A)$, only depends on the original class h^p.

All the statements of §§3.3, 3.4 can be repeated in the case of relative cohomology by replacing the word "map" by "map of pairs" (6).

3.6 Remark. We can say something more precise about the properties of the differential form $d\psi$ of §3.5, when A is a closed submanifold of X. In §I.6.4 we defined ψ to be equal to $\pi \cdot \mu^*\varphi$ in a tubular neighbourhood of A, where μ was the retraction of this tubular neighbourhood onto A, and π was a $\mathscr{C}^\infty$ function which was equal to 1 *in a neighbourhood of* A. Therefore, in this latter neighbourhood,

$$d\psi = d\mu^*\varphi = \mu^* d\varphi = 0,$$

so that *the support of* $d\psi$ *does not meet* A, which is a stronger property than the property $d\psi|A = 0$ which was established in §3.5.

4 De Rham duality

4.1

Let us make the definition that $C^*(X)$ is the $\mathbb{Z}$-dual of $C_*(X)$ (mentioned in §3.1) more precise. A *cochain* (with coefficients in $\mathbb{Z}$) is a *linear* map $\varphi : C_*(X) \to \mathbb{Z}$. To every chain γ, φ associates an integer denoted $\langle \varphi \mid \gamma \rangle$. We will say that the cochain φ is of *degree* p if $\langle \varphi \mid \gamma \rangle = 0$ for every chain of dimension different from p. The coboundary homomorphism δ is *defined* by

$$\langle \delta\varphi \mid \gamma \rangle = \langle \varphi \mid \partial\gamma \rangle \cdot$$

This relation implies that $\langle \varphi \mid \gamma \rangle$ is zero whenever γ is a cycle and φ is a coboundary, or γ is a boundary and φ a cocycle. As a result, $\langle \varphi \mid \gamma \rangle$ only depends on the homology class of γ and on the cohomology class of φ, and defines a *bilinear form*

$$\langle \mid \rangle : H^p(X) \otimes H_p(X) \longrightarrow \mathbb{Z}.$$

We can then pose the following algebraic problem: since $C^*(X)$ is the $\mathbb{Z}$-dual of $C_*(X)$ by $\langle \mid \rangle$, can we deduce that $H^*(X)$ is the $\mathbb{Z}$-dual of $H_*(X)$ by $\langle \mid \rangle$?

The answer is *yes, modulo torsion.* If, instead of taking coefficients in $\mathbb{Z}$, we took them in a *field*, then we avoid any torsion problems and $H^*(X)$ *is the vector space which is dual to* $H_*(X)$.

4.2

The duality of §4.1 becomes particularly interesting in the light of *de Rham's theorem*. This states that we can replace $C^*(X)$ by $\Omega(X)$, δ by d, and $\langle \mid \rangle$ by integration in the definition of cohomology $H^*(X)$ with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (the formula $\langle \delta\varphi \mid \gamma \rangle = \langle \varphi \mid \partial\gamma \rangle$ then becomes *Stokes' formula*).

We can therefore state the

Theorem (De Rham duality). *The bilinear form given by "integration" identifies the cohomology of differential forms $H^p(X)$ with the space of all linear functions on $H_p(X)$.*

Corollary. *If φ is a closed differential form whose integral vanishes along any cycle, then φ is an exact differential form.* [*As a linear function on $H_p(X)$, φ is zero, and therefore de Rham's duality theorem asserts that its cohomology class is zero.*]

4.3

The considerations of §4.1 can obviously be extended to *relative* homology and cohomology [the relative *cochains* of (X, A) are those whose restriction to A is zero, i.e., the cochains φ such that $\langle \varphi \mid \gamma \rangle = 0$, $\forall\, \gamma \in i_* C_*(A)$, which shows that $C^*(X, A)$ is indeed the dual of $C_*(X, A) = C_*(X)/i_* C_*(A)$].

De Rham's theorem can also be extended so that *the relative cohomology $H^p(X, A)$ of differential forms is identified, via the bilinear form given by "integration", with the space of all linear functions on $H_p(X, A)$.*

4.4 Transposition of ∂_* and δ^*

We immediately deduce from the transposition formula $\langle \delta\varphi \mid \gamma \rangle = \langle \varphi \mid \partial\gamma \rangle$ (Stokes' formula), the following transposition formula

$$\langle \delta^* k \mid h \rangle = \langle k \mid \partial_* h \rangle,$$

where ∂_* and δ^* are the homomorphisms defined in §§2.11 and 3.5:

$$h \in H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A);$$
$$H^p(X, A) \xleftarrow{\delta^*} H^{p-1}(A) \ni k.$$

5 Families of supports. Poincaré's isomorphism and duality

5.1

We call a *family of supports* in a topological space[7] X, a set Φ of closed subsets of X satisfying the following three properties:

$$(\Phi_1) \qquad A,\, B \in \Phi \Longrightarrow A \cup B \in \Phi;$$

$$(\Phi_2) \qquad \left.\begin{array}{c} A \in \Phi \\ B \text{ closed} \subset A \end{array}\right\} \Longrightarrow B \in \Phi;$$

(Φ_3) [8] every $A \in \Phi$ has a neighbourhood belonging to the family Φ.

Examples. The family F of all closed sets, and the family c of all compact sets, are both families of supports.

If X is a *subspace* of the space Y, and Ψ is a family of supports of Y, one can check that the families

$$\Psi|X = \{A \subset X :\ A \in \Psi\};^{9}$$
$$\Psi \cap X = \{A = B \cap X :\ B \in \Psi\}$$

are families of supports on X.

In particular, $c_Y|X = c_X$, $F_Y \cap X = F_X$.

5.2 Cohomology with supports in Φ

We will not give the definition of the *support of a cochain*, which is rather delicate. Let us only recall, in the case of manifolds, that the *support of a differential form* φ (abbreviated to $\operatorname{supp}\varphi$) is the smallest closed set where $\varphi(x) \neq 0$.

The supports of cochains, as for differential forms, are closed sets which satisfy the following properties:

$$(\text{supp}^1) \qquad \operatorname{supp}(\varphi + \psi) \subset \operatorname{supp}\varphi \cup \operatorname{supp}\psi,$$

$$(\text{supp}^2) \qquad \operatorname{supp}\delta\varphi \subset \operatorname{supp}\varphi;$$

$$(\text{supp}^3) \qquad \forall\, f : X \longrightarrow Y \text{ and } \forall\, \psi \in C^*(Y), \quad \operatorname{supp} f^*\psi \subset f^{-1}(\operatorname{supp}\psi).$$

[7] Throughout this section, "topological spaces" will be assumed to be *locally compact and paracompact* (3).

[8] Condition (Φ_3) does not play an essential role, and in any case, will only be used in §§5.6 and 6.5.

[9] In order for this family to satisfy condition (Φ_3), the subspace X must be assumed to be "*locally closed*" (i.e., the intersection of an open and a closed subset of Y).

Let Φ be a family of supports on X. Let $C^*({}^\Phi X)$ denote the *group of cochains with supports in the family* Φ:

$$C^*({}^\Phi X) = \{\varphi \in C^*(X) : \operatorname{supp}\varphi \in \Phi\}\cdot$$

This is a *group*, because of (Φ_1) (supp^1), and it is *stable under the coboundary map* δ, because of (Φ_2) (supp^2). We can therefore define its *cohomology*,

$$H^*({}^\Phi X) = \text{cohomology of } C^*({}^\Phi X).$$

A continuous map

$$f : {}^\Phi X \longrightarrow {}^\Psi Y$$

of topological spaces equipped with families of supports will be called "*cohomologically admissible*" if $f^{-1}(\Psi) \subset \Phi$. It follows from (supp^3) that any such map induces a homomorphism

$$f^* : C^*({}^\Psi Y) \longrightarrow C^*({}^\Phi X),$$

and therefore a homomorphism

$$f^* : H^*({}^\Psi Y) \longrightarrow H^*({}^\Phi X),$$

and the correspondence $f \rightsquigarrow f^*$ is a *contravariant functor* (defined on the category of admissible maps). In particular, if we take the family of all *closed* sets to be the family of supports, all continuous maps are cohomologically admissible (since the inverse image of a closed set under a continuous map is closed). That is why *cohomology with closed supports* $H^*({}^F X)$, which is simply "ordinary cohomology" $H^*(X)$, has a privileged status.

5.3 Homology with supports in Φ

We call the *support of a chain element* $[\sigma]$ (abbreviated to supp$[\sigma]$) the image of the map σ in X. It is a *compact* subset of X (the image of the compact set Δ under the continuous map σ).

Let us modify the definition of chains in §1.2 by declaring that the formal linear combination

$$\gamma = \sum_i n_i[\sigma_i]$$

will no longer necessarily be finite, but *locally finite*: in other words, every point x has a neighbourhood U_x which only meets a finite number of the supports of the elements $[\sigma_i]$. It is easy to deduce that the set

$$\operatorname{supp}\gamma = \bigcup_i \operatorname{supp}[\sigma_i]$$

is *closed*, which is called the "support of the chain γ".[10]

These closed sets satisfy the following properties:

$$(\mathrm{supp}_1) \qquad \operatorname{supp}(\gamma+\gamma') = \operatorname{supp}\gamma \cup \operatorname{supp}\gamma',$$

$$(\mathrm{supp}_2) \qquad \operatorname{supp}\partial\gamma \subset \operatorname{supp}\gamma.$$

Let us denote by $C_*({}_\Phi X)$ the *group of chains with supports in the family Φ*:

$$C_*({}_\Phi X) = \{\gamma \in C_*(X):\ \operatorname{supp}\gamma \in \Phi\}\cdot$$

This is indeed a *group*, because of (Φ_1) (supp_1), and it is stable under the *boundary map* ∂, because of (Φ_2) (supp_2). We can therefore define its *homology*,

$$H_*({}_\Phi X) = \text{homology of } C_*({}_\Phi X).$$

A continuous map

$$f : {}_\Phi X \longrightarrow {}_\Psi Y$$

of topological spaces equipped with families of supports will be called "*homologically admissible*" if:

(f_1) f is "Φ-proper", i.e.:
$\forall$ compact sets $K \subset Y$ and $\forall A \in \Phi$, $f^{-1}(K) \cap A$ is compact.

(f_2) $f(\Phi) \subset \Psi$

It is easy to see that condition (f_1) is equivalent to the following:

(f_1') For every locally finite family $\{A_i \subset X\}$ such that $\bigcup_i A_i \in \Phi$, the family $\{f(A_i)\}$ is locally finite in Y.

This condition therefore allows us to define the image $f_*\gamma$ of a chain γ with support in Φ, and this image will have its support in Ψ because of condition (f_2) along with the following obvious property:

$$(\mathrm{supp}_3) \qquad \forall f : X \longrightarrow Y \quad \text{and} \quad \forall \gamma \in C_*(X), \qquad \operatorname{supp} f_*\gamma = f(\operatorname{supp}\gamma).$$

Any such admissible map therefore induces a homomorphism

$$f_* : H_*({}_\Psi X) \longrightarrow H_*({}_\Psi Y)$$

and the correspondence $f \rightsquigarrow f_*$ is a *covariant functor* on the category of admissible maps.

In particular, if we take the family of *compact sets* to be the family of supports, every continuous map is homologically admissible, since:

1° every closed subset of a compact set is compact,
2° the image of a compact set under a continuous map is compact.

This homology with compact supports $H_*({}_c X)$ coincides incidentally with "ordinary" homology $H_*(X)$ (because every locally finite family of subsets of a compact set is *finite*).

[10] With the definition of chains given in §1.2 we run into the difficulty that the set $\bigcup \operatorname{supp}[\sigma_i]$ will in general *depend* on the choice of representative for γ. This difficulty does not occur for the more modern definitions of chains.

5.4 Poincaré's isomorphism[11]

In an *oriented* manifold X of dimension n, there exists, *for every family of supports* Φ, a *canonical isomorphism*

$$\boxed{H_p({}_{\Phi}X) \approx H^{n-p}({}^{\Phi}X)}$$ [12]

between homology and cohomology with the same coefficients.

We will not give the construction of this isomorphism here, since its properties will appear naturally due to the notion of a "*current*", which will be introduced in section 6. In much the same way as "distributions" are a generalization of functions, "currents" are a generalization of differential forms, and happen at the same time to be a generalization of chains.

5.5 Intersection index. Poincaré duality

Let us consider the bilinear form

$$\langle \mid \rangle : H^p(X) \otimes H_p(X) \longrightarrow \mathbb{Z}$$

defined in §1. Poincaré's isomorphism maps the cohomology group $H^p(X)$ (with closed supports) to the homology group $H_{n-p}({}_F X)$, and maps the bilinear form above into the bilinear form

$$\langle \mid \rangle : H_{n-p}({}_F X) \otimes H_p(X) \longrightarrow \mathbb{Z}$$

which can be interpreted as the "*intersection index*" of cycles. In section 7 we will see the precise geometric meaning of this intersection index.

De Rham's duality theorem becomes "*Poincaré duality*":

The homology group with closed supports $H_{n-p}({}_F X)$ is, modulo torsion, the $\mathbb{Z}$-dual to $H_p(X)$ for the "intersection" bilinear form.

Special case. If the manifold X is compact, the family of closed sets coincides with the family of compact sets, and Poincaré duality becomes a duality between the "ordinary" homology groups $H_p(X)$ and $H_{n-p}(X)$.

Corollary. *For every compact connected orientable manifold X of dimension n, $H_n(X) \approx \mathbb{Z}$.*

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[11] Many authors call this isomorphism "Poincaré duality".

[12] **Corollary.** *All homology and cohomology groups vanish in dimension $> n$.*

5.6 Leray coboundary

Let X be a differentiable manifold, and let S be a *closed* submanifold of codimension r. We have already noted (remark 3.6) that in the construction of §3.5 which was used to define the homomorphism δ^*, we associated to every closed differential form φ of S, a form $d\psi$ of X *whose support does not meet* S. On the other hand, the support of ψ can be taken to be an arbitrarily small neighbourhood of ψ (by taking the tubular neighbourhood of §I.6.4 to be sufficiently small). Therefore, if $\operatorname{supp}\varphi \in \Phi|S$, where Φ is a family of supports of X, we can use axiom (Φ_3) for families of supports to ensure that $\operatorname{supp}\psi \in \Phi$, and in this way, $\operatorname{supp} d\psi \in \Phi|X - S$.

From this, we deduce a homomorphism (7)

$$\delta^* : H^p(^{\Phi}|S) \longrightarrow H^{p+1}(^{\Phi}|X - S).$$

If the manifolds X and S are oriented, Poincaré's isomorphism applied to δ^* gives a homomorphism between homology groups

$$\delta^* : H_{q-r}({}_{\Phi}|S) \longrightarrow H_{q-1}({}_{\Phi}|X - S) \qquad (q = n - p)$$

which is called the "*Leray coboundary*". It can be interpreted geometrically in the following way:

the retraction $\mu : V \rightarrow S$ makes the tubular neighbourhood V into a "fibre bundle" (cf. chap. IV, §2) with base S whose fibres are spheres of dimension r. The boundary $\dot{V}$ of V is therefore fibred by $(r-1)$-spheres. The Leray coboundary map "inflates" $(q-r)$-cycles of S inside $X - S$ by fibring them by these $(r-1)$-spheres.

Special case. $\Phi = c$ (the family of compact sets of X).

$\Phi|S$ and $\Phi|X - S$ are therefore simply the families of compact sets of S and $X - S$, and the Leray coboundary can be written

$$\delta^* : H_{q-r}(S) \longrightarrow H_{q-1}(X - S).$$

6 Currents

Throughout this section, X is an *oriented* differentiable manifold of dimension n.

6.1

A *current* j on X is a continuous linear form[13] on $\Omega(^cX)$. In particular:

[13] The word "continuous" should be understood in the following way: if $\varphi_i (i = 1, 2, \dots)$ are $\mathscr{C}^\infty$ differential forms whose supports are all contained in the same compact set (equipped with a chart) and whose coefficients, calculated in this chart, tend uniformly to zero, along with all their derivatives, as $i \to \infty$, then $j[\varphi_i] \to \infty$ with i.

- *Every chain γ defines a current*

$$\gamma[\varphi] = \langle \varphi \mid \gamma \rangle = \int_\gamma \varphi;$$

- *every differential form ω defines a current*

$$\omega[\varphi] = \int_X \omega \wedge \varphi.$$

If $j[\varphi]$ only differs from zero on the forms φ of degree p, we will say that the *dimension* of the current j is p, or that its *degree* is $n-p$. We will denote by $J_p(X) = J^{n-p}(X)$ the vector space of currents of dimension p (of degree $n-p$).

6.2 The support of a current

We will say that the current j is *zero on an open set D if* $j[\varphi] = 0$ for every differential form φ whose (compact) support is contained in D.

The following theorem shows that the notion of a current is *local.*

Theorem. *If $j = 0$ in a neighbourhood U_x of every point $x \in D$, then $j = 0$ on D.*

To see this, let $\{\pi_x\}_{x \in D}$ be a locally finite partition of unity which is subordinate to the covering $\{U_x\}_{x \in D}$. For every form φ supported on D,

$$j[\varphi] = j\Big[\sum_x \pi_x \varphi\Big] = \sum_x j\,[\pi_x \varphi] = 0$$

(since the support of φ is compact, the sum only has a *finite* number of non-zero terms, which then enables us to use the linearity of j).

Definition. The complement of the largest open set on which $j = 0$ is called the *support* of j (the previous theorem gives a meaning to the notion of the "largest open set where $j = 0$").

If j is a form (resp. chain), this definition coincides with the notion of the support of a form (resp. chain).

Remark. Clearly, the symbol $j[\varphi]$ can even be defined for forms φ with non-compact support, provided that $\operatorname{supp} j \cap \operatorname{supp} \varphi$ is compact. More generally, we can even define the "convergence" of the symbol $j[\varphi]$, in an analogous way to the convergence of an integral (§I.7.6).

Currents with supports in a family Φ. We will let

$$J_p({}_\Phi X) = J^{n-p}({}_\Phi X)$$

denote the vector space of currents of dimension p with supports in a family Φ. Let $f : {}_\Phi X \to {}_\Psi Y$ be a differentiable map, which is *homologically admissible* (§5.3). We will show that it induces a linear map $f_* : J_p({}_\Phi X) \to J_p({}_\Psi Y)$, which is the transpose of the linear map f^* on differential forms (the correspondence $f \rightsquigarrow f_*$ will therefore be a covariant functor).

Since f is Φ-proper (property (f_1) of §5.3), the closed set

$$\operatorname{supp} j \cap \operatorname{supp} f^*\psi \subset \operatorname{supp} j \cap f^{-1}(\operatorname{supp} \psi)$$

is *compact* for every current j with supports in Φ and every form ψ with compact support. We can therefore define the current f_*j by the formula $(f_*j)[\psi] = j[f^*\psi]$. On the other hand, one can easily check that

$$\operatorname{supp} f_*j \subset f(\operatorname{supp} j),$$

so that, by property (f_2) of §5.3, $\operatorname{supp} f_*j \in \Psi$.

6.3 The boundary and differential of a current

The current ∂j is defined by

$$\boxed{\partial j[\varphi] = j[d\varphi].}$$

By Stokes' formula, this definition coincides with the usual definition of the boundary when the current j is equal to a chain.

The current dj is defined by

$$\boxed{dj = w\,\partial j,}$$

where w is the linear operator which multiplies a current of degree p by $(-)^p$.

Let us check that this definition coincides with the usual definition for the differential when the current j is equal to a form ω:

$$\begin{aligned} d\omega[\varphi] &= \int_X d\omega \wedge \varphi = \int_X d(\omega \wedge \varphi) - w\omega \wedge d\varphi \\ &= -\int_X w\omega \wedge d\varphi = -w\omega[\varphi] = -\partial w\omega[\varphi] = w\partial\omega[\varphi]. \end{aligned}$$

6.4 Homology of currents

This is defined in the obvious way from the operator ∂ or d (which amounts to the same thing), and we will write it

$$H_pJ({}_\Phi X) = H^{n-p}J({}_\Phi X)$$

for currents of dimension p (of degree $n-p$) with supports in the family Φ.

6.5 Homologies between currents and differential forms

We saw in §6.1 that every differential form defines a current: we therefore have a canonical map

$$\Omega({}^{\Phi}X) \longrightarrow J({}_{\Phi}X).$$

Theorem. *This canonical map induces an isomorphism of cohomology groups:*

$$\boxed{H^p({}^{\Phi}X) \approx H^p J({}_{\Phi}X).}$$

The idea of the proof is to construct a "*regularization operator*" R_ε, which commutes with ∂, and maps every current to a $\mathscr{C}^\infty$ form (by "smoothing" it by a $\mathscr{C}^\infty$ function, in the same way that one regularizes a distribution to produce a function). We can take the parameter ε to be small enough that the operator R_ε modifies the currents by as small an amount as we desire. In particular, the support of $\mathrm{R}_\varepsilon j$ can be taken in an arbitrarily small neighbourhood of the support of j, and can therefore be made to belong to the same family of supports [by axiom (Φ_3) for families of supports].

6.6 Homologies between currents and chains

Since every chain defines a current (§6.1), we have a canonical map

$$C_*({}_{\Phi}X) \longrightarrow J({}_{\Phi}X).$$

Theorem. *This canonical map induces an isomorphism of homology groups:*

$$H_p({}_{\Phi}X) \approx H_p J({}_{\Phi}X).$$

By composing this isomorphism with the isomorphism of §6.5, we obtain Poincaré's isomorphism (§5.4)*.*

6.7 A useful example: the current defined by a closed oriented submanifold

A *closed oriented submanifold* S of dimension p (with or without boundary) obviously defines a current of dimension p:

$$S[\varphi] = \int_S \varphi|S$$

[we required that S be closed to ensure that $\operatorname{supp}(\varphi|S)$ is compact if $\operatorname{supp}\varphi$ is].

The *support* of such a current obviously coincides with the set S.

By Stokes' formula as given in §I.7.7, its *boundary* is the current defined by the oriented submanifold ∂S. In particular, if the submanifold S has no boundary, it defines a "*closed*" current, and therefore a *homology class* on the space X. However, the converse, that every homology class of X can be represented by a submanifold, is in general *false* (it depends on the dimension). This difficult question forms part of the theory of "cobordism" due to Thom [35].

7 Intersection indices

7.1

In §5.5 we defined the *intersection index*[14] $\langle \gamma' \mid \gamma \rangle$ of two cycles (or rather their homology classes). More generally, we will try to define, at least in certain cases, the intersection index $\langle k \mid j \rangle$ of two *currents* j and k.

If k is equal to a differential form, we will set $\langle k \mid j \rangle = j[k]$: this is nothing other than the integral $\int_j k$ if j is equal to a chain, whereas if j is also equal to a differential form,

$$\langle k \mid j \rangle = j[k] = \int_X j \wedge k.$$

Now let j and k be two arbitrary currents. We will say that the symbol $\langle k \mid j \rangle$ has a meaning if, whatever the choice of regularizations R_ε and R'_ε (cf. §6.5), $\langle \mathrm{R}'_\varepsilon k \mid \mathrm{R}_\varepsilon j \rangle$ tends to a limit when $\varepsilon \to 0$, and we define $\langle k \mid j \rangle$ to be equal to this limit.

If $\operatorname{supp} j \cap \operatorname{supp} k$ is compact, and if one of the symbols $\langle k \mid \partial j \rangle$, $\langle dk \mid j \rangle$ has a meaning, then the other also has a meaning and they are equal:

$$\langle \mathrm{R}'k \mid \mathrm{R}\,\partial j \rangle = \langle \mathrm{R}'k \mid \partial \mathrm{R} j \rangle = \langle d\mathrm{R}'k \mid \mathrm{R} j \rangle = \langle \mathrm{R}'dk \mid \mathrm{R} j \rangle.$$

In particular, $\langle k \mid j \rangle = 0$ every time that one of the currents is closed and the other is homologous to 0. Therefore if j and k are two closed currents ($\partial j = \partial k = 0$), $\langle k \mid j \rangle$ *only depends on the homology classes of j and k.*

7.2

We say that the current j is $\mathscr{C}^\infty$ *at a point x* if it is equal, in a neighbourhood U_x of x, to a $\mathscr{C}^\infty$ differential form ω (i.e., if $j - \omega = 0$ in U_x, in the sense of §6.2). The set of points where j is $\mathscr{C}^\infty$ is clearly an open set. Its complement is called the "*singular support*" of the current j. If $\operatorname{supp} j \cap \operatorname{supp} k$ is compact, and if the singular supports of j and k do not meet each other, the symbol $\langle k \mid j \rangle$ clearly has a meaning, since this locally reduces to the case where one of the two currents is a $\mathscr{C}^\infty$ form. In fact, it is possible to prove a stronger theorem (due to de Rham), using certain properties of regularizations:

Theorem. *The symbol $\langle k \mid j \rangle$ has a meaning if* $\operatorname{supp} j \cap \operatorname{supp} k$ *is compact, and if the singular support of each current does not meet the singular support of the boundary of the other current.*

[14] Also called the "*Kronecker index*".

7.3

Suppose that the currents j and k are defined by *closed oriented submanifolds with or without boundaries*, S_1 and S_2 (cf. §6.7). By observing that the support of a $\mathscr{C}^\infty$ form cannot be of measure zero, one sees that the singular supports of these two currents are simply the sets S_1 and S_2 (unless the dimension of one of these submanifolds is n).

The preceding theorem can be stated as follows: *in order for the symbol $\langle S_2 \mid S_1 \rangle$ to have a meaning, it is enough for $S_1 \cap S_2$ to be compact and*

$$\partial S_1 \cap S_2 = S_1 \cap \partial S_2 = \varnothing .$$

We shall give a simple method for calculating this symbol in the case where the submanifolds S_1 and S_2 (of dimension p and $n-p$) *intersect transversally* (10).

Their intersection is therefore a set of isolated points $x^{(1)}, x^{(2)}, \dots, x^{(N)}$, which is finite because $S_1 \cap S_2$ is compact. Let us say that the orientations of S_1 and S_2 *match* at the point $x^{(i)}$ if the system of tangent vectors at the point $x^{(i)}$:

$$(V_1,\, V_2, \dots, V_p,\, W_1,\, W_2, \dots, W_{n-p}) ,$$

obtained by concatenating a system of indicators $(V_1,\, V_2, \dots, V_p)$ for S_1, and a system of indicators $(W_1,\, W_2, \dots, W_{n-p})$ for S_2, is a system of indicators for the orientation on X.

Theorem. $\langle S_2 \mid S_1 \rangle = N^+ - N^-$, *where N^+ (resp. N^-) is the number of points $x^{(i)}$ where the orientations of S_1 and S_2 match (resp. do not match).*

It clearly suffices to prove it in *a neighbourhood* of a point $x^{(i)}$. In other words, it suffices to check that if

$$\begin{array}{lll} X = \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}, & \text{oriented by the vectors} & \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dots, \dfrac{\partial}{\partial x_n} \right); \\ S_1 = \mathbb{R}^p, & \text{"} \qquad \text{"} \qquad \text{"} & \left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_p} \right); \\ S_2 = \mathbb{R}^{n-p}, & \text{"} \qquad \text{"} \qquad \text{"} & \left(\dfrac{\partial}{\partial x_{p+1}}, \dots, \dfrac{\partial}{\partial x_n} \right), \end{array}$$

then $\langle S_2 \mid S_1 \rangle = 1$. This can be checked directly, by using a regularization construction (which we have not given here).

7.4 Example. Let X be the n-dimensional complex sphere (of real dimension $2n$), and let S be the compact submanifold of dimension n defined by the real part of this sphere. We will give X the canonical orientation coming from its complex analytic structure, and give S an arbitrary orientation. Then the intersection index of S with itself is given by *É. Cartan's formula* [2]:

$$\langle S \mid S \rangle = \begin{cases} \quad 0 & \text{if } n \text{ is odd;} \\ 2(-)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. S defines a *closed* current since it has no boundary. By §7.1, the intersection index is not changed if we replace the submanifold S by a submanifold S' which defines a homologous current:

$$\langle S' \mid S\rangle = \langle S \mid S\rangle.$$

We can then ensure that S' intersects S transversally, in order to be able to use §7.3.

It is convenient to observe that the complex sphere X is homeomorphic to the "*tangent bundle*" of the real sphere S, i.e., the space of all tangent vectors at every point of S. If

$$z_k = x_k + iy_k \qquad (k = 0,\, 1,\, 2, \ldots, n)$$

are the coordinates of $\mathbb{C}^{n+1}$, the equation of X in $\mathbb{C}^{n+1}$ can be written:

$$z \cdot z = \sum_{k=0}^{n} z_k^2 = 1, \qquad \text{i.e.,} \quad \begin{cases} x \cdot x - y \cdot y = 1, \\ \qquad\quad x \cdot y = 0. \end{cases}$$

Now map any point $z \in X$ to the point with coordinates $x_k/\sqrt{1 + y \cdot y}$ in the real Euclidean space $\mathbb{R}^{n+1}$, and to the vector with components y_k which is based at this point. This clearly defines a homeomorphism from X onto the space of tangent vectors of the unit sphere $S \subset \mathbb{R}^{n+1}$. In this way, every vector field which is tangent to the unit sphere represents a submanifold $S' \subset X$, which is clearly homologous[15] to the original submanifold S (represented by the zero vector field). Thus, we are reduced, by §7.3, to study the points where such a vector field vanishes. Let us take, for example, the vector field $y(x)$ defined by the coordinates (in the ambient Euclidean space)

$$\begin{aligned} y_k(x) &= \lambda x_k x_0 \qquad (k = 1,\, 2, \ldots, n), \\ y_0(x) &= \lambda \left(x_0^2 - 1\right), \end{aligned}$$

where λ is an arbitrary positive parameter (Fig. II.2).

This vector field is indeed tangent to the sphere, since

$$x \cdot y(x) = \lambda x_0 (x \cdot x - 1) = 0.$$

Furthermore, it only vanishes at the two poles $P^{\pm}$ of the sphere

$$P^{\pm} : x_1 = x_2 = \cdots = x_n = 0, \qquad x_0 = \pm 1.$$

Near these two poles, the coordinates $x_1, x_2, \ldots, x_n$ provide a local chart of the sphere, and the field $y(x)$ is approximately

$$y_k(x) \simeq \pm\lambda x_k \quad \text{near the point } P^{\pm}.$$

[15] In fact, every vector field on a manifold is obviously homotopic to the zero vector field: it suffices to multiply it by a scalar λ which tends to zero.

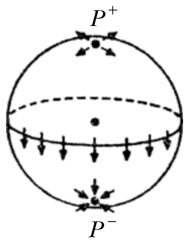


Fig. II.2.

As a result, if the manifold S is described near P^+ by the system of indicators

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n},$$

the manifold S' will be described by the system of indicators

$$\frac{\partial}{\partial x_1} + \lambda\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} + \lambda\frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial x_n} + \lambda\frac{\partial}{\partial y_n}$$

(which tends to the first system when $\lambda \to 0$, which shows that we have oriented S' in the correct way) (cf. Fig. II.3).

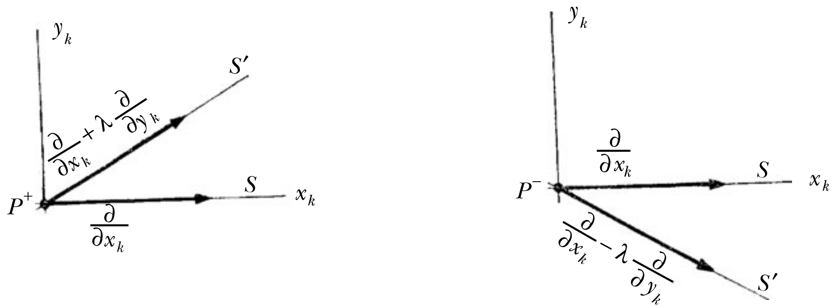


Fig. II.3.

We then see, by recalling that the canonical orientation of a complex manifold is given by the system of indicators

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n},$$

that the orientations of S and S' at the point P^+ match, up to the sign $(-)^{n(n-1)/2}$. A similar argument at the point P^- gives the same result multiplied by $(-)^n$ [because of the minus sign in the equation $y_k(x) = -\lambda x_k$].

We therefore obtain the result stated earlier:

$$\langle S' \mid S \rangle = (-)^{n(n-1)/2} \left(1 + (-)^n\right).$$

Remarks.

(i) The result for odd n could be obtained directly by using the skew symmetry of the intersection index

$$\langle S' \mid S \rangle = (-)^{\dim S \, \dim S'} \langle S \mid S' \rangle,$$

which in this case gives:

$$\langle S \mid S \rangle = (-)^{n^2} \langle S \mid S \rangle \cdot$$

(ii) A corollary of the result for even n is the well-known theorem which states that a continuous vector field on an even-dimensional sphere necessarily vanishes somewhere.

III

Leray's theory of residues

This theory generalizes Cauchy's theory of residues to complex analytic manifolds. Throughout this chapter, X will denote a complex analytic manifold, and, up to section 4, S will denote a closed analytic submanifold of complex codimension 1. A local equation for S (with non-zero gradient) in a neighbourhood of one of its points y, will be denoted $s_y(x) = 0$. If S has a global equation, this will be denoted by $s(x) = 0$. In general, the subscript y will mean that a function (or a differential form) is only defined in a neighbourhood $U_y \subset X$ of the point $y \in S$.

1 Division and derivatives of differential forms

1.1 Lemma (on the division of forms, local version). *If ω_y is a differential form (which is regular in the neighbourhood under consideration) such that $ds_y \wedge \omega_y = 0$, then we can choose a form ψ_y near y, which is also regular, such that*

$$\omega_y = ds_y \wedge \psi_y.$$

The restriction $\psi_y|S$ of this form only depends on ω_y and s_y, and is denoted

$$\psi_y|S = \left.\frac{\omega_y}{ds_y}\right|_S .$$

If ω_y is holomorphic at the point y, ψ_y can be chosen to be holomorphic at that point, and as a result, $\psi_y|S$ is holomorphic.

Proof. This is obvious on choosing local coordinates $\xi_1, \xi_2, \ldots, \xi_n$ on X such that $s_y(x) = \xi_1$, and by expanding the differential forms in the basis spanned by $d\xi_1, d\xi_2, \ldots, d\xi_n$. The uniqueness of the restriction $\psi_y|S$ follows by observing that if $\omega_y = 0$ (i.e., $d\xi_1 \wedge \psi_y = 0$), then ψ_y must have $d\xi_1$ as a factor, and $d\xi_1|S = 0$. □

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_3,

1.1 Lemma (on the division of forms, global version). *If S has a global equation $s(x) = 0$, and if ω is a regular differential form on X satisfying $ds \wedge \omega = 0$, then we can choose a regular form ψ on X, such that*

$$\omega = ds \wedge \psi.$$

We denote the restriction of this form by $\psi|S = \omega/ds|_S$. This restriction only depends on ω and s, and is holomorphic if ω is holomorphic.

Proof. We construct forms ψ_y locally as previously, and superimpose them using a partition of unity (note that if ω is holomorphic, there is no reason for the form ψ constructed in this way to be holomorphic. However, $\psi|S$ is holomorphic, by the local lemma). □

1.2 Derivative of a form

In contrast to the division of forms, the notion of the derivative of a form with respect to s supposes that s is a *global* equation.

Let ω be a *closed regular form on* X, such that $ds \wedge \omega = 0$. By lemma 1.1, there exists a form ω_1 which is regular on X, such that $\omega = ds \wedge \omega_1$; since ω is closed,

$$ds \wedge d\omega_1 = -d\omega = 0 \overset{1.1}{\Longrightarrow} \exists\, \omega_2 \,:\, d\omega_1 = ds \wedge \omega_2;$$

but

$$ds \wedge d\omega_2 = -d(d\omega_1) = 0 \overset{1.1}{\Longrightarrow} \exists\, \omega_3,\ \cdots \overset{1.1}{\Longrightarrow} \exists\, \omega_\alpha,:\ d\omega_{\alpha-1} = ds \wedge \omega_\alpha.$$

Every step in this construction involves a choice (the choice of ψ in lemma 1.1). Nonetheless,

1.3 Proposition. *The cohomology class of $\omega_\alpha|S$ on S is well-defined given ω and s.*

Proof. It suffices to show that if $\omega = 0$, $\omega_\alpha|S$ is cohomologous to zero. Now,

$$\omega = ds \wedge \omega_1 = 0 \overset{1.1}{\Longrightarrow} \exists\, \chi_1 \,:\, \omega_1 = ds \wedge \chi_1, \qquad d\omega_1 = -ds \wedge d\chi_1;$$

and

$$\begin{aligned} d\omega_1 = ds \wedge \omega_2 &\Longrightarrow ds \wedge (\omega_2 + d\chi_1) = 0 \\ &\overset{1.1}{\Longrightarrow} \exists\, \chi_2 \,:\, \omega_2 = -d\chi_1 + ds \wedge \chi_2, \qquad d\omega_2 = -ds \wedge d\chi_2; \end{aligned}$$

and

$$\begin{aligned} d\omega_2 = ds \wedge \omega_3 &\Longrightarrow ds \wedge (\omega_3 + d\chi_2) = 0,\ \dots \\ &\overset{1.1}{\Longrightarrow} \exists\, \chi_\alpha \,:\, \omega_\alpha = -d\chi_{\alpha-1} + ds \wedge \chi_\alpha. \end{aligned}$$

As a result,

$$\omega_\alpha|S = d\left(-\chi_{\alpha-1}|S\right).$$

□

1.4 Notation. The *cohomology class* of $\omega_\alpha|S$ in S is denoted $d^{\alpha-1}\omega/ds^\alpha\big|_S$:

$$\omega_\alpha|S \in \left.\frac{d^{\alpha-1}\omega}{ds^\alpha}\right|_S \in H^*(S).$$

This notation suggests formally writing

$$\omega_\alpha|S \in \left.\frac{d^{\alpha-1}(ds\wedge\omega_1)}{ds^\alpha}\right|_S = \left.\frac{d^{\alpha-1}\omega_1}{ds^{\alpha-1}}\right|_S,$$

and, likewise, if $\omega = d\omega'$ (still satisfying $ds\wedge\omega = 0$),

$$\omega_\alpha|S \in \left.\frac{d^{\alpha-1}(d\omega')}{ds^\alpha}\right|_S = \left.\frac{d^{\alpha}\omega'}{ds^\alpha}\right|_S.$$

Leray shows that these conventions are coherent, and that the "derivatives" defined in this way satisfy the usual rules for the calculation of derivatives (Leibniz' formula).

1.5 Remark. All the above could have been done in the case of *real* manifolds in the same way (where S is of *real* codimension 1). In this case, Gelfand and Chilov [14] have introduced the notion of division and derivation of forms, in order to define the "Dirac measure" $\delta(s)$ and its derivatives. Let $X = \mathbb{R}^n$, and let $S \subset \mathbb{R}^n$ be an oriented submanifold defined by the equation $s(x) = 0$. The action of $\delta(s)$ on a test function $f(x)$ is defined by

$$\delta(s)[f] = \int_S \left.\frac{\omega}{ds}\right|_S,$$

where $\omega = f(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ (note that the dimension of S is indeed equal to the degree of $\omega/ds|_S$, which is $n-1$). In the same way, the α-fold derivative of $\delta(s)$ is defined by

$$\delta^{(\alpha)}(s)[f] = \int_S \left.\frac{d^\alpha\omega}{ds^{\alpha+1}}\right|_S.$$

Physicists use these concepts all the time.

2 The residue theorem in the case of a simple pole

2.1 Lemma (Residue form). *Let φ be a closed regular form on $X - S$, with a "polar singularity of order 1" along S, i.e., $\forall\, y \in S$, $s_y\varphi$ is the restriction of a form ω_y, which is regular in a neighbourhood of y, to $X - S$. Then there exist regular forms ψ_y and θ_y near y, such that*

$$\boxed{\varphi = \frac{ds_y}{s_y}\wedge\psi_y + \theta_y,}$$

where $\psi_y|S$ is a closed form which only depends on φ. It is called the "residue form" of φ, and is denoted

$$\boxed{\operatorname{res}[\varphi] = \left.\frac{s_y\varphi}{ds_y}\right|_S \cdot}$$

This form is holomorphic if φ is meromorphic.

Proof. (i) *Existence of ψ_y and θ_y.* The form $d\omega_y$ is regular, and $ds_y \wedge d\omega_y = 0$ (since $d\omega_y = ds_y \wedge \varphi$ on $X - S$). Thus, by lemma 1.1 (local version), there exists a regular form θ_y near y such that $d\omega_y = ds_y \wedge \theta_y$. The form $s_y(\varphi - \theta_y)$ is regular, and

$$ds_y \wedge s_y(\varphi - \theta_y) = 0,$$

so there exists near y a regular form ψ_y such that

$$s_y(\varphi - \theta_y) = ds_y \wedge \psi_y.$$

(ii) *$\psi_y|S$ only depends on φ and s_y.* It is enough to prove that if $\varphi = 0$, then $\psi_y|S = 0$. But,

$$\begin{aligned}
\frac{ds_y}{s_y} \wedge \psi_y + \theta_y = 0 \implies \qquad & ds_y \wedge \theta_y = 0 \overset{1.1}{\implies} \exists\, \theta'_y,\ \theta_y = ds_y \wedge \theta'_y \\
&\implies ds_y \wedge \big(\psi_y + s_y\theta'_y\big) = 0 \overset{1.1}{\implies} \exists\, \theta''_y,\ \psi_y + s_y\theta'_y = ds_y \wedge \theta''_y \\
&\implies \psi_y|S = 0.
\end{aligned}$$

(iii) *$\psi_y|S$ does not depend on s_y.* Let $s^*_y = 0$ be another local equation for S near y.

If $\varphi = (ds_y/s_y) \wedge \psi_y + \theta_y$, we can write in the same way

$$\varphi = \frac{ds^*_y}{s^*_y} \wedge \psi_y + d\Big(\log \frac{s_y}{s^*_y}\Big) \wedge \psi_y + \theta_y.$$

But since s_y/s^*_y is a non-zero holomorphic function near y, its logarithm is regular, and so we can set

$$\begin{aligned}
\theta^*_y &= \theta_y + d\Big(\log \frac{s_y}{s^*_y}\Big) \wedge \psi_y, \\
\psi^*_y &= \psi_y.
\end{aligned}$$

(iv) *$\psi_y|S$ is closed.* Since φ is closed,

$$\frac{ds_y}{s_y} \wedge d\psi_y - d\theta_y = 0;$$

but, by the argument in (ii), this implies that $d\psi_y|S = 0$.

(v) *Justification of the notation* $\operatorname{res}[\varphi] = s_y\varphi/ds_y|_S$. In the case where $ds_y \wedge \varphi = 0$ near y, we can do the construction in (i) with $\theta_y = 0$, and $\psi_y|S$ coincides with the form $\omega_y/ds_y|_S$ defined by lemma 1.1. □

2.2 Leray coboundary

The definition in §II.5.6 provides, in the particular case which interests us here ($\mathrm{codim}_{\mathbb{R}}\, S = 2$), a homomorphism:

$$\delta^* : H_p(S) \longrightarrow H_{p+1}(X - S).$$

We will interpret this geometrically in some more detail.

Let $\overline{V}$ be a closed tubular neighbourhood of the submanifold S, equipped with a retraction $\mu : \overline{V} \to S$ which makes it into a fibre bundle, with fibres given by the unit disk D in the complex plane (we can think of the complex-valued function s_y, restricted to each fibre $\mu^{-1}(y')$, as giving an isomorphism of this fibre for y' near y onto a disk in the complex plane, and this isomorphism gives the fibre a canonical *orientation*). Let σ be a chain element of dimension p of S, which, by abuse, we can take to be an oriented polyhedron which is *embedded* in S. The set $\mu^{-1}(\sigma)$ is the product of the disk D with the polyhedron σ, and defines in X the chain element, or "cell",

$$\mu^*\sigma = D \otimes \sigma$$

(the oriented product of the oriented cells D and σ). Its boundary[1] is

$$\partial\mu^*\sigma = \partial\,(D \otimes \sigma) = \partial D \otimes \sigma + D \otimes \partial\sigma.$$

Let $\delta_\mu : C_p(S) \to C_{p+1}(X - S)$ be the homomorphism which maps a chain element σ to the chain $\partial D \otimes \sigma$. Intuitively, this homomorphism "inflates" the chains of S in $X - S$ by fibring them with the circle ∂D. It anti-commutes with the boundary homomorphism, because

$$\partial\delta_\mu\,\sigma = \partial\,(\partial D \otimes \sigma) = -\partial D \otimes \partial\sigma = -\delta_\mu\,\partial\sigma.$$

It therefore descends to homology, and defines a homomorphism

$$\delta_* : H_p(S) \longrightarrow H_{p+1}(X - S)$$

which coincides with δ^* up to a sign. In fact,

$$\delta^* = w\delta_*$$

(to understand this sign, cf. the formula of §II.6.3, which relates the boundary and the derivative of a current). In the sequel, the homomorphism δ_* will be called the Leray coboundary, and will simply be denoted by δ.

[1] The general formula for the boundary of a product is:

$$\partial(\sigma \otimes \tau) = \partial\sigma \otimes \tau + (-)^{\dim\sigma}\sigma \otimes \partial\tau.$$

2.3 Transposition of ∂ and δ

Let σ be a cycle in S (of dimension p), and let τ be a relative cycle of (X, S) (of dimension $2n-p-1$). By composing σ with Poincaré's isomorphism, and by applying Stokes' formula, one sees that

$$\langle \delta^* \sigma \mid \tau \rangle = \langle \sigma \mid \partial \tau \rangle.$$

In other words,

$$(-)^{p+1} \langle \delta_* \sigma \mid \tau \rangle = \langle \sigma \mid \partial \tau \rangle.$$

This relation can be illustrated by the following geometric argument (Fig. III.1).

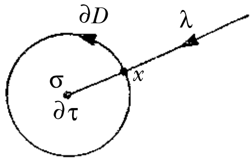


Fig. III.1.

Let us suppose that the cycles σ and τ are homologous[2] to submanifolds, which will enable us to use definition II.7.3 for intersection indices. Suppose also that the submanifold τ intersects S transversally. We can choose the metric in such a way that τ is a locus of geodesics which are orthogonal to S. Therefore, if the retraction μ is defined using this metric, and if $\delta_\mu \sigma$ and τ intersect transversally at the point x, then σ and $\partial\tau$ will intersect transversally at the point $\mu(x)$. Let λ be the geodesic which joins x to $\mu(x)$, oriented in this direction. At the point x, the submanifolds $\delta_\mu \sigma$ and τ can be locally represented by the oriented products

$$\begin{aligned} \delta_\mu \sigma &= \partial D \otimes \sigma, \\ \tau &= \lambda \otimes \partial \tau, \end{aligned}$$

which gives the orientation rule

$$\begin{aligned} \varepsilon_X(\tau, \delta_\mu \sigma) = \varepsilon_X(\lambda, \partial\tau, \partial D, \sigma) &= (-)^p \varepsilon_X(\lambda, \partial D, \partial \tau, \sigma) \\ &= (-)^p \varepsilon_{\mathbb{C}}(\lambda, \partial D) \varepsilon_S(\partial\tau, \sigma) = (-)^{p+1} \varepsilon_S(\partial \tau, \sigma). \end{aligned}$$

2.4 Theorem (Residue theorem). *If γ is a cycle on S (with compact support) and φ is a closed differential form on $X - S$ with a polar singularity along S of order* 1*, we have the "residue formula":*

$$\boxed{\int_{\delta\gamma} \varphi = 2\pi i \int_\gamma \operatorname{res}[\varphi],}$$

[2] As currents.

where res$[\varphi]$ *is the residue form of* φ*, as defined in* §2.1*, and* $\delta\gamma$ *is the Leray coboundary of* γ*, as defined in* §2.2*.*

Proof. Consider a family of representatives $\delta_{\mu_\varepsilon}\gamma$ for the homology class $\delta\gamma$, defined by retractions $\mu_\varepsilon : \overline{V}_\varepsilon \to S$, where the "radius" ε of the tubular neighbourhood $\overline{V}_\varepsilon$ tends to zero. We have

$$\int_{\delta\gamma} \varphi = \int_{\delta_{\mu_\varepsilon}\gamma} \varphi \quad \text{for all } \varepsilon, \qquad \text{so} \quad = \lim_{\varepsilon\to 0} \int_{\delta_{\mu_\varepsilon}\gamma} \varphi.$$

Let σ be one of the chain elements of γ, with support contained in the domain U of a local chart on X. For sufficiently small ε, the support of $\delta_{\mu_\varepsilon}\sigma$ will be in U, so that if the coordinate ξ_1 of the chart under consideration represents the local equation of S, we will have

$$\varphi = \frac{d\xi_1}{\xi_1} \wedge \psi + \theta \quad \text{in } U,$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\delta_{\mu_\varepsilon}\sigma} \varphi &= \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon \otimes \sigma} \frac{d\xi_1}{\xi_1} \wedge \psi \\ &= \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon} \frac{d\xi_1}{\xi_1} \int_\sigma \psi|S = 2\pi i \int_\sigma \psi|S = 2\pi i \int_\sigma \mathrm{res}[\varphi]. \qquad \square\end{aligned}$$

2.5 Corollary. *The cohomology class of* res$[\varphi]$ *in* S *only depends on the cohomology class of* φ *in* $X - S$*. It is called the "residue class" of* φ*, and is denoted* Res$[\varphi]$*.*

Proof. If φ is cohomologous to zero in $X - S$, its integral over every cycle of $X - S$ (and in particular over every cycle of the form $\delta\gamma$) is zero. Therefore $\int_\gamma \mathrm{res}[\varphi]$ is zero for any cycle γ in S. But by de Rham's theorem, this means that res$[\varphi]$ is cohomologous to zero in S. □

3 The residue theorem in the case of a multiple pole

3.1 Theorem (Existence of the "residue class" (J. Leray)). *Every closed regular form* φ *on* $X - S$ *is cohomologous in* $X - S$ *to a form* $\widetilde{\varphi}$ *which has a simple pole on* S*.*

Definition. The cohomology class of res$[\widetilde{\varphi}]$, which is defined uniquely by corollary 2.5, is called the *residue class* of φ and is denoted by Res$[\varphi]$.

We will admit this theorem without proof, but in §3.3 will indicate a case where the construction of a form $\widetilde{\varphi}$ is easy to do explicitly.

3.2 Theorem (Residue theorem). *If γ is a cycle on S (with compact support), and φ is a "closed" form on $X - S$, we have the "residue formula":*

$$\boxed{\int_{\delta\gamma} \varphi = 2\pi i \int_{\gamma} \mathrm{Res}[\varphi],}$$

where $\mathrm{Res}[\varphi]$ *is the residue class of φ, whose existence is guaranteed by theorem 3.1.*

This is an immediate consequence of §§2.4 and 3.1.

3.3 Construction of the residue class when S has a global equation

Hypotheses. *S has a global equation* $s(x) = 0$, and *the form φ has a polar singularity on S of order α* ($\alpha > 1$), i.e., $s^\alpha \varphi$ can be extended to give a form ω which is regular on X.

The problem is to construct a form $\widetilde{\varphi}$ which is cohomologous to φ in $X - S$, and has a polar singularity of order 1 on S.

By an argument based on §2.1 (i) (but which uses the *global* lemma 1.1), one can prove the existence of forms ψ and θ, which are regular on X, such that

$$\varphi = \frac{ds}{s^\alpha} \wedge \psi + \frac{\theta}{s^{\alpha-1}} \cdot$$

Therefore,

$$\varphi = d\left(-\frac{1}{\alpha - 1}\frac{\psi}{s^{\alpha-1}}\right) + \frac{1}{s^{\alpha-1}}\left(\frac{d\psi}{\alpha - 1} + \theta\right).$$

In other words, φ is cohomologous in $X - S$ to the form

$$\frac{\theta_1}{s^{\alpha-1}} = \frac{1}{s^{\alpha-1}}\left(\frac{d\psi}{\alpha - 1} + \theta\right)$$

which has a polar singularity on S of order $\alpha - 1$. The form $\widetilde{\varphi}$ is constructed in this way by induction on α.

3.4 Relation with the notion of the derivative of a form

Let us add the hypothesis $ds \wedge \varphi = 0$ to the hypotheses of §3.3. In this case, the form $\omega = s^\alpha \varphi$ is regular and *closed* in X: $d\omega = \alpha s^{\alpha-1} ds \wedge \varphi = 0$, and obviously $ds \wedge \omega = 0$. Therefore the hypotheses of §1.2 are satisfied, and it is obvious from the construction of §3.3 that the residue class $\mathrm{Res}[\varphi]$ coincides with $1/(\alpha - 1)!$ times the class $d^{\alpha-1}\omega/ds^\alpha|_S$ defined in §1.2:

$$\mathrm{Res}[\varphi] = \frac{1}{(\alpha - 1)!} \left.\frac{d^{\alpha-1}\omega}{ds^\alpha}\right|_S .$$

In the general case, Leray takes this relation to be the *definition* of the symbol $d^{\alpha-1}\omega/ds^\alpha|_S$, and shows that this convention is compatible with the usual operations on derivatives (cf. §1.4).

4 Composed residues

4.1

Let $S_1, S_2, \ldots, S_m$, be m closed analytic submanifolds of X of complex codimension 1, *which intersect in general position*, in other words, near every intersection point

$$y \in S^I = \bigcap_{i \in I} S_i \qquad (I \subset \{1, 2, \ldots, m\}),$$

the differentials ds_{iy} $(i \in I)$ are linearly independent (s_{iy} denotes the local equation of S_i near y). Therefore, every intersection S^I is a *complex analytic submanifold* of S^J for $J \subset I$, of complex codimension $|I| - |j|$, by §I.4.5. Thus, by replacing the pair (X, S) of the preceding sections by appropriately chosen pairs, we can define a sequence of "*residue*" homomorphisms:

$$\begin{aligned}
&H^p(X - S_1 \cup \cdots \cup S_m)\\
&\xrightarrow{\mathrm{Res}_1} H^{p-1}(S_1 - S_2 \cup \cdots \cup S_m)\\
&\xrightarrow{\mathrm{Res}_2} H^{p-2}(S_1 \cap S_2 - S_3 \cup \cdots \cup S_m) \xrightarrow{\mathrm{Res}_3} \cdots \xrightarrow{\mathrm{Res}_m} H^{p-m}(S_1 \cap \cdots \cap S_m)
\end{aligned}$$

as well as a sequence of "*Leray coboundary*" homomorphisms:

$$\begin{aligned}
&H_p(X - S_1 \cup \cdots \cup S_m)\\
&\xleftarrow{\delta_1} H_{p-1}(S_1 - S_2 \cup \cdots \cup S_m)\\
&\xleftarrow{\delta_2} H_{p-2}(S_1 \cap S_2 - S_3 \cup \cdots \cup S_m) \xleftarrow{\delta_3} \cdots \xleftarrow{\delta_m} H_{p-m}(S_1 \cap \cdots \cap S_m).
\end{aligned}$$

These homomorphisms can be composed to give homomorphisms

$$\begin{aligned}
H_p(X - S_1 \cup \cdots \cup S_m) &\xrightarrow{\mathrm{Res}^m} H_{p-m}(S_1 \cup \cdots \cup S_m),\\
H_p(X - S_1 \cup \cdots \cup S_m) &\xleftarrow{\delta^m} H_{p-m}(S_1 \cap \cdots \cap S_m)
\end{aligned}$$

which are "transposed" by the *composed residue formula*:

$$\int_{\delta^m \gamma} \varphi = (2\pi i)^m \int_\gamma \mathrm{Res}^m[\varphi]$$

(it suffices to apply the ordinary residue formula m times).

4.2 Skew symmetry of the composed coboundary

The cycle

$$\delta^m \gamma = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_m \gamma$$

is obtained by "fibring" γ by an oriented product of m unit circles[3]

$$\partial D_1 \otimes \partial D_2 \otimes \cdots \otimes \partial D_m.$$

As a result, if η is a permutation of $\{1, 2, \dots, m\}$,

$$\delta_{\eta_1} \circ \delta_{\eta_2} \circ \cdots \circ \delta_{\eta_m} \gamma = (-)^{\eta} \delta_1 \circ \delta_2 \circ \cdots \circ \delta_m \gamma,$$

where $(-)^{\eta}$ is the signature of the permutation. Stated differently, the composed coboundary is a *skew-symmetric* operation with respect to the indices $1, 2, \dots, m$, and the same is therefore true for the composed residue.

4.3 Calculation of composed residues

The composed residue of a form φ with polar singularities of order $\alpha_1, \alpha_2, \dots, \alpha_m$ on $S_1, S_2, \dots, S_m$ is written

$$\mathrm{Res}^m[\varphi] = \frac{1}{(\alpha_1 - 1)! \cdots (\alpha_m - 1)!} \left. \frac{d^{\alpha_1 + \cdots + \alpha_m - m} \omega}{ds_1^{\alpha_1} \wedge ds_2^{\alpha_2} \wedge \cdots \wedge ds_m^{\alpha_m}} \right|_{S_1 \cap \cdots \cap S_m},$$

where $\omega = s_1^{\alpha_1} s_2^{\alpha_2} \dots s_m^{\alpha_m} \varphi$, and can be calculated by analogous methods to those discussed in previous sections.

If $s_1, s_2, \dots, s_m$ are *global* equations, and if ω is a regular differential form near $S_1, S_2, \dots, S_m$, such that

$$ds_1 \wedge \cdots \wedge ds_m \wedge d\omega = 0,$$

we set

$$\left. \frac{\partial^{\alpha_1 + \cdots + \alpha_m} \omega}{\partial s_1^{\alpha_1} \dots \partial s_m^{\alpha_m}} \right|_S = \left. \frac{d^{\alpha_1 + \cdots + \alpha_m} [ds_1 \wedge \cdots \wedge ds_m \wedge \omega]}{ds_1^{1+\alpha_1} \wedge \cdots \wedge ds_m^{1+\alpha_m}} \right|_S .$$

The "*partial derivatives*" defined in this way satisfy all the usual rules for partial derivatives (symmetry with respect to the indices, Leibnitz' formula, and the change of variable formula).

5 Generalization to relative homology

All the concepts in the preceding sections can easily be generalized to relative homology and cohomology. If S and S' are two complex analytic submanifolds

[3] Note that the composed coboundary δ^m has nothing to do with the coboundary

$$\delta^* : H_p(S_1 \cap \cdots \cap S_m) \longrightarrow H_{p+2m-1}(X - S_1 \cap \cdots \cap S_m)$$

obtained by applying the definition of §II.5.6 to the submanifold

$$S_1 \cap \cdots \cap S_m \subset X.$$

of codimension 1, *which intersect in general position*, one defines the Leray coboundary

$$\delta : H_p(S, S') \longrightarrow H_{p+1}(X - S, S')$$

using the same construction as in §2.2, except that one chooses the retraction μ in such a way that $\mu(S') \subset S'$ (to do this, one can take a metric near S such that S' is a locus of geodesics which are orthogonal to S).

In the same way, there is a"residue" homomorphism:

$$\mathrm{Res} : H_{p+1}(X - S, S') \longrightarrow H_p(S, S').$$

The residue formula is still the same:

$$\int_{\delta\gamma} \varphi = 2\pi i \int_{\gamma} \mathrm{Res}[\varphi],$$

where, this time, γ is a *relative* cycle on (S, S'), and φ is a closed differential form on $X - S$ satisfying $\varphi | S' = 0$.

IV

Thom's isotopy theorem

1 Ambient isotopy[1]

Recall that a homeomorphism $g : (X, A) \approx (Y, B)$ between two pairs of topological spaces consists of a homeomorphism $g : X \approx Y$ and two subspaces $A \subset X$, $B \subset Y$ such that $g|A : A \approx B$.

1.1

A homeomorphism

$$g : (X, S_0) \approx (X, S_1)$$

is called an *ambient isotopy*[2] of S_0 in X if there exists in X a family of subspaces $\{S_\tau\}_{\tau \in [0,1]}$ which interpolate S_0 and S_1, and a family of *homeomorphisms* $g_\tau : (X, S_0) \approx (X, S_\tau)$ which depend continuously[3] on τ such that $g_0 = \text{identity}$, $g_1 = g$.

The subspaces S_0 and S_1 are therefore called "*isotopic in* X". Let us emphasize that an ambient isotopy consists solely of the homeomorphism g: the "interpolating families" $\{S_\tau\}$ and $\{g_\tau\}$ are assumed to exist, but are not specified; the extra data giving these families will be called a *realization* of the ambient isotopy. The homeomorphisms which realize an ambient isotopy can be conveniently represented on a diagram such as figure IV.1.

1.2

We will say that two ambient isotopies

$$g, h : (X, S_0) \approx (X, S_1)$$

[1] Even though this concept is sufficiently interesting in its own right, one can first read section 1 of chapter VI for some motivation.

[2] More precisely: an "ambient isotopy from S_0 to S_1 inside X".

[3] This means that the map $G : X \times [0, 1] \to X$ defined by $G(x, \tau) = g_\tau(x)$ is continuous.

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_4,

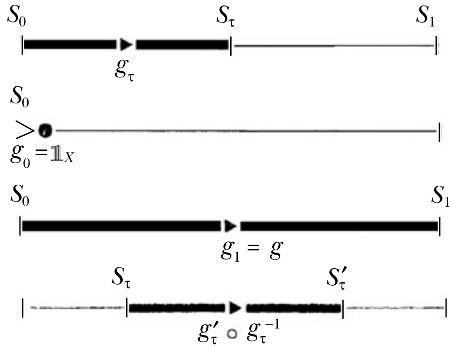


Fig. IV.1.

are *equivalent* (in short, $g \equiv h$) if the *same* family of interpolating spaces $\{S_\tau\}$ can be used to realize both of them. The relation $g \equiv h$ is indeed an *equivalence relation*: the reflexivity and symmetry are obvious; and the transitivity is too by the following

1.3 Lemma. *Let* $g \overset{S_\tau}{\equiv} h$ *be two ambient isotopies realized by*

$$g_\tau, h_\tau : (X, S_0) \approx (X, S_\tau).$$

If another family of interpolating spaces S'_τ *can be used to realize* g*, then it can also be used to realize* h*.*

To see this, let $g'_\tau : (X, S_0) \approx (X, S'_\tau)$ denote the second realization of g. It follows that h can be realized by

$$h'_\tau = g'_\tau \circ g_\tau^{-1} \circ h_\tau : (X, S_0) \approx (X, S'_\tau).$$

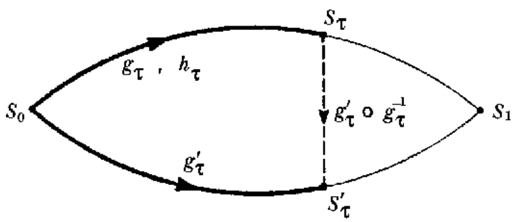


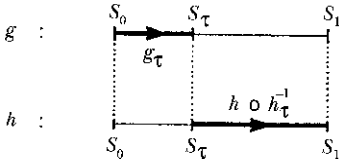
Fig. IV.2.

Definition. The equivalence classes for the relation $\equiv$ will be called *ambient isotopy classes*.

1.4 Theorem. *Two equivalent ambient isotopies are homotopic.*

If $g \overset{S_\tau}{\equiv} h$, the family of maps $\gamma_\tau = h \circ h_\tau^{-1} \circ g_\tau$ realizes the homotopy:

$$g \simeq h : (X, S_0) \approx (X, S_1).$$



1.5 Corollary. *Two ambient isotopies which are equivalent induce the same isomorphisms between homology groups:*

$$\begin{aligned} g \equiv h \Longrightarrow g_* = h_* : \qquad & H_*({}_\Phi X, S_0) \approx H_*({}_\Phi X, S_1) \\ \textit{or} \quad & H_*({}_\Phi|X - S_0) \approx H_*({}_\Phi|X - S_1) \\ \textit{or} \quad & H_*({}_\Phi|S_0) \approx H_*({}_\Phi|S_1) \end{aligned}$$

for every family of supports Φ which is invariant under homeomorphisms.

1.6 Example. We set:
$X = \mathbb{C}$ (the complex plane);
$S_0 = S_1$, the set consisting of two points $x = +1$, $x = -1$;
g, the identity map;
h, the "symmetry map centred at O": $h(x) = -x$.

The homeomorphisms g and h: $(X, S_0) \approx (X, S_0)$ are ambient isotopies of S_0 in X. They can be realized by

$$S_\tau = S_0, \qquad \text{with} \quad g_\tau = \mathbb{1}_X$$

and $S'_\tau = \{x = e^{i\pi\tau}, x = -e^{i\pi\tau}\}$, with

$$h_\tau = \text{rotation by the angle } \pi\tau : \; h_\tau(x) = e^{i\pi\tau}x.$$

These isotopies are not equivalent, since

$$g_* : H_0(S_0) \longrightarrow H_0(S_0)$$

is the identity automorphism, whereas

$$h_* : H_0(S_0) \longrightarrow H_0(S_0)$$

is the automorphism which swaps the two generators of $H_0(S_0)$ (since h interchanges the two points $x = +1$, $x = -1$).

2 Fiber bundles

2.1

A *fibre bundle* will be a topological space Y equipped with a *projection* (a continuous surjective map)

$$\pi : Y \longrightarrow T$$

onto a space T which is called the *base*, such that for all $t \in T$, the space $Y_t = \pi^{-1}(t)$ is homeomorphic to a given space X which is called the *fibre*.

Y_t is called the "fibre over t".

Trivial example. The product space $X \times T$, equipped with the obvious projection onto T.

Fiber bundle map. A "fibre bundle map" $f : Y \to Y'$ from a fibre bundle Y to a fibre bundle Y' *with the same base* T is a continuous map which "respects the projection map", i.e., such that the diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & T & \end{tikzcd}$$

commutes. In other words, f maps the fibre Y_i into the fibre Y_i' over the same point t. In particular, if $Y = X \times T$ and $Y' = X' \times T$, giving a fibre bundle map f is the same as giving a continuous map $f_i : X \to X'$ which depends continuously on t.

The inverse image of a bundle. Given a continuous map $k : T' \to T$, we naturally associate, to every bundle Y with fibre X and base T, a bundle Y' *with the same fibre* X but with base T', along with a commutative diagram:

$$\begin{tikzcd} Y' \arrow[r, "\widetilde{k}"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ T' \arrow[r, "k"] & T \end{tikzcd}$$

This bundle will be called the *inverse image of the bundle* Y *by the mapk* [in short, $Y' = k^{-1}(Y)$].

The construction is done in the following way: the space Y' is the subspace of $Y \times T'$ consisting of the points (y, t') such that $\pi(y) = k(t')$, and the maps π', $\widetilde{k}$ are defined by

$$\pi'(y, t') = t', \qquad \widetilde{k}(y, t') = y.$$

Special case. If T' is a subspace of T, and k is the natural injection, then $k^{-1}(Y)$ is identified with the subspace of points in Y which project onto T'. We say that $k^{-1}(Y)$ is the *restriction of the bundle* Y *to* T' [abbreviated to $k^{-1}(Y) = Y|T'$].

Sections of a bundle. A "section" σ of the fibre bundle Y is a continuous map $\sigma : T \to Y$ such that $\pi \circ \sigma = \mathbb{1}_T$. In other words, a section maps every point t of the base to a point on the fibre Y_t, continuously with respect to t.

2.2 The trivial bundle

A fibre bundle Y is called "*trivial*" if there exists a *homeomorphism of bundles* $f : Y \approx X \times T$, i.e., a homeomorphism f such that the diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X \times T \arrow[dl, "\pi'"] \\ & T & \end{tikzcd}$$

commutes.

The homeomorphism f will be called a "*trivialization of a bundle*" of the bundle Y.

Examples of non-trivial bundles:

2.3

Let $Y = T = \mathbb{C} - \{0\}$ denote the punctured complex plane, and $\pi : Y \to T$ the map $\pi(y) = y^2$. The fibre Y_t is the set of two points

$$y = \sqrt{t} \qquad \text{and} \qquad y = -\sqrt{t}.$$

X is therefore a space consisting of two points.
Y is not homeomorphic to $X \times T$, since Y is connected, but $X \times T$ is not.

2.4

Let T be a differentiable manifold of dimension n, and let Y be its "tangent bundle" (cf. Example II.7.4).

Here, Y_t is the vector space which is tangent to the variety at the point t, and the fibre X is therefore a vector space of dimension n. A section of Y is a continuous vector field on T. It is easy to see that the tangent bundle of an *even-dimensional sphere* is not trivial. We have already seen in §II.7.4 that two sections of this tangent bundle *always* intersect, which would not happen if the bundle were trivial.

2.5 Locally trivial fibre bundles

The fibre bundle Y is called "locally trivial" if every point t of the base has a neighbourhood Θ_t, such that $Y|\Theta_t$ is a trivial bundle (8).

Often, when one speaks of a fibre bundle, the phrase "locally trivial" is implied. All the examples that we have given are locally trivial. A classical theorem states that *every locally trivial fibre bundle over a contractible base is trivial.*[4]

Another useful (and obvious) property is that the inverse image of a [locally] trivial fibre bundle is [locally] trivial.

2.6 Pairs of fibre bundles

The notions above naturally extend to the case of pairs of topological spaces (cf. §II.2.11). A *pair of fibre bundles* with base T will be a pair (Y, S) equipped with a projection $\pi : Y \to T$ such that for every $t \in T$, the pair (Y_t, S_t) is homeomorphic to a given pair (X, S_0) [we have set $S_t = \pi^{-1}(t) \cap S$]. The pair of fibre bundles (Y, S) will be called *trivial* if there exists a homeomorphism of fibre bundles $f : Y \approx X \times T$ such that

$$f|S : S \approx S_0 \times T.$$

In the same way, one can define a locally trivial pair of fibre bundles.

Isotopy class of a path. Let $(X \times T, S)$ be a *locally trivial* pair of fibre bundles with base T, and let $\lambda : [0,1] \to T$ be a path in T. The pair of fibre bundles $\lambda^{-1}(X \times T, S)$ is locally trivial, with contractible base $[0,1]$, and is therefore trivial.

A trivialization of this pair is equivalent to giving a continuous family of homeomorphisms

$$f_\tau : X \approx X \qquad \text{such that} \quad f_\tau | S_{\lambda(\tau)} : S_{\lambda(\tau)} \approx S_0,$$

and therefore an *ambient isotopy*

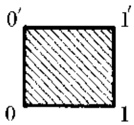
$$g = f_1^{-1} \circ f_0 : \big(X, S_{\lambda(0)}\big) \approx \big(X, S_{\lambda(1)}\big)$$

whose class obviously does not depend on the choice of trivialization, but only on the path.

In fact, let us show that *this class only depends on the homotopy class of the path*: if λ and λ' are two paths with the same endpoints

$$t_0 = \lambda(0) = \lambda'(0) \quad \text{and} \quad t_1 = \lambda(1) = \lambda'(1),$$

a homotopy (with fixed endpoints) between these two paths is a continuous map A from the square

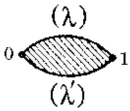


[4] The base must be assumed to be locally compact and paracompact.

to T, such that

$$A|[0,1] = \lambda, \quad A|[0',1'] = \lambda', \quad A([0,0']) = t_0, \quad A([1,1']) = t_1.$$

After identifying all the points in the segment $[0,0']$ on the one hand, and those in $[1,1']$ on the other hand, we are reduced to a continuous map A_*, from the diagram



into T. The fibre bundle $A_*^{-1}(X \times T, S)$ is locally trivial, with contractible base, and is therefore trivial. A trivialization $f_\tau : X \approx X$ of this fibre bundle

defines, by restriction, a trivialization of $\lambda^{-1}(X \times T, S)$ and $\lambda'^{-1}(X \times T, S)$. Therefore the homeomorphism $g = f_1^{-1} \circ f_0$ is an ambient isotopy associated *to two paths* λ and λ'.

2.7 Example. Let $X = \mathbb{C}$, $T = \mathbb{C} - \{0\}$, and let S be the submanifold of $X \times T$ with the equation $x^2 - t = 0$. The pair of fibre bundles $(X \times T, S)$ is locally trivial, and its fibre is the pair $(X, S_1 = \{x = 1, x = -1\})$. One can associate the ambient isotopy h of example 1.6 to the path $\lambda : [0,1] \to T$ defined by $\lambda(\tau) = e^{i\pi\tau}$.

3 Stratified sets

3.0 Example. Consider the surface S defined in Euclidean space $\mathbb{R}^3$, by the equation

$$x^2 - y^2 z = 0 \quad \text{(Fig. IV.3)}$$

and decompose it into *manifolds* as follows:

$$S = A_1^2 \cup A_2^2 \cup A_1^1 \cup A_1^2 \cup A^0,$$

where

$$\begin{aligned} &A_1^2 = S \cap \{y > 0\}, && A_2^2 = S \cap \{y < 0\}, \\ &A_1^1 = \{x = y = 0, z > 0\}, && A_2^1 = \{x = y = 0, z < 0\}, \\ &A^0 = \text{the origin}. \end{aligned}$$

We say that we have *stratified* the set S, and the manifolds A are called the *strata*. The *boundary* of a stratum A is the set $\partial A = \overline{A} - A$. In the particular case considered above,

$$\partial A_1^2 = \partial A_2^2 = A_1^1 \cup A^0, \quad \partial A_1^1 = \partial A_2^1 = A^0, \quad \partial A^0 = \varnothing.$$

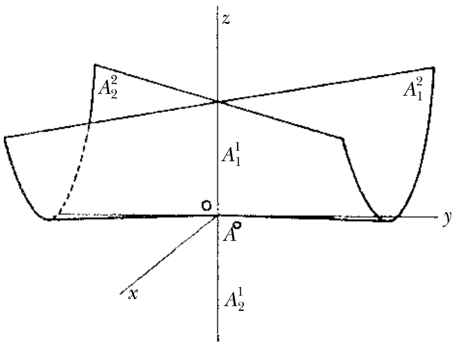


Fig. IV.3.

3.1 Primary stratification

Let Y be a differentiable manifold of dimension p, and let S be a *closed* subset of Y. We say that S is equipped with a *primary stratification* if S has a *family of nested closed sets*

$$S = S^{p_1} \supset S^{p_2} \supset \cdots \supset S^{p_\lambda} \supset \cdots, \qquad p_1 > p_2 > \cdots > p_\lambda > \cdots$$

such that

1° ("*strata*") $S^{p_\lambda} - S^{p_{\lambda+1}}$ is a *differentiable submanifold* of Y, of dimension p_λ, whose connected components $A_i^{p_\lambda}$ (assumed to be finite in number) are called the *strata*;[5]

2° ("*boundary property*") the boundary $\partial A = \overline{A} - A$ of a stratum A is a union of strata of dimension strictly less than those of A.

3.2 Example. If, in example 3.0, we had taken the subvarieties A_1^2, A_2^2 and $A^1 = \{x = y = 0\}$ as "strata", the boundary property would not have held.

3.3 Primary stratification of a complex analytic set

Let σ denote the operation which to any complex analytic set S associates the analytic set $\sigma(S)$ of its singular points. Starting with an analytic set S, one can construct, by iterating the three operations σ, $\cap$ and "decomposition into irreducible components", a family of *irreducible* analytic sets S_α. We set

$$A_\alpha = S_\alpha - \bigcup_{\beta \prec \alpha} S_\beta,$$

where $\beta \prec \alpha$ means that S_β is strictly smaller than S_α. A_α is, by construction, an *analytic submanifold.* Since S_α is irreducible, A_α is *connected*, and $\overline{A}_\alpha = S_\alpha$ (cf. [1], § 44 C), so that

$$\partial A_\alpha = \bigcup_{\beta \prec \alpha} S_\beta.$$

[5] We use the convention that a manifold of negative dimension is empty.

The boundary property is therefore satisfied. In this way, we have canonically constructed a primary stratification of S.

3.4 Example. Let us repeat example 3.0, but in $\mathbb{C}^3$ rather than $\mathbb{R}^3$.
The canonical construction above gives the family of sets

$$\{S^2 = S,\ S^1 = \sigma(S) = \text{the } z \text{ axis}\}.$$

The canonical stratification is therefore

$$A^2 = S - S^1, \quad A^1 = S^1.$$

3.5 Example. Let $S = S_1 \cup S_2 \cup \cdots \cup S_m$ be a union of *closed submanifolds in general position.* The canonical construction above gives the following family of non-singular analytic sets

$$S_i, \quad S_i \cap S_j, \quad S_i \cap S_j \cap S_k, \ \ldots,$$ [6]

and hence the stratification

$$A_i = S_i - \bigcup_{j \neq i} (S_i \cap S_j),$$
$$A_{ij} = S_i \cap S_j - \bigcup_{k \neq i \neq j} S_i \cap S_j \cap S_k, \ \ldots$$

3.6 Regular incidence

The stratum A is said to be incident to the stratum B (abbreviated to $A \prec B$) if $A \subset \partial B$. The family of strata B to which A is incident is called the *star of* A.

In order to define a *stratification*, we will impose some extra conditions on the primary stratification, called "*regular incidence*". These conditions will be local, so that we can assume that the ambient manifold Y is a Euclidean space (9). We will denote by $T_a(A)$ the hyperplane (in the Euclidean space Y) which is tangent to the stratum A at a point $a \in A$. We will say that the strata $A \prec B$ are *regularly incident at a point a of A* if the two following conditions, called *Whitney's conditions A and B*, are satisfied as the point b of B varies in a neighbourhood of a:

A. the angle between $T_b(B)$ and $T_a(A)$ tends to zero with $|b - a|$;

[6] More precisely, one must consider the connected components ($\Longleftrightarrow$ irreducible components) of these varieties.

B. if $r : B \to A$ is a local retraction[7] of B onto A, the angle between $T_b(B)$ and the vector $\overrightarrow{b, r(b)}$ tends to zero with $|b - a|$.[8]

Examples.

3.7

Consider example 3.2 (or rather 3.4, if we wish the boundary condition to be satisfied, but this is of no great importance). At a point along the axis Oy, the tangent plane to the stratum A^2 is horizontal, and therefore orthogonal to the stratum A^1: Whitney's condition A is not satisfied at the origin by the strata $A^1 \prec A^2$.

3.8

Let $S \subset \mathbb{R}^3$ be the surface defined by the equation

$$y(y - z^2) + x^2 = 0 \quad [\text{“the tapered cone”, Fig. IV.4}]$$

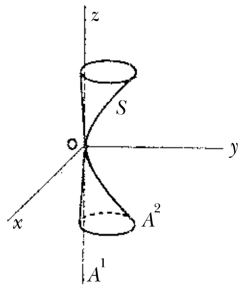


Fig. IV.4.

with the strata

$$A^1 = \text{the } z \text{ axis}, \quad A^2 = S - A^1.$$

Let $r : A^2 \to A^1$ be the orthogonal projection of A^2 onto the axis A^1. When the point b tends to zero along the parabola $A_2 \cap \{x = 0\}$, the tangent plane $T_b(A^2)$ becomes orthogonal to the direction $\overrightarrow{b, r(b)}$: Whitney's condition B is not satisfied at the origin by the strata $A^1 \prec A^2$.

[7] In other words a map (defined in a neighbourhood of a) $r : B \to A$ such that $r \cup \mathbb{1}_A : B \cup A \to A$ is continuous. Strictly speaking, it is therefore $r \cup \mathbb{1}_A$ which is a “retraction”. In order to construct r, one can “foliate” Y with parallel hyperplanes which are transversal to A near a, and define $r(b)$ to be the point of intersection between A and the hyperplane passing through b.

[8] Observe that if A is satisfied, condition B for any one retraction r implies B for every retraction.

3.9

Let $S \subset \mathbb{R}^3$ be the cylinder (Fig. IV.5) defined in semi-polar coordinates (ρ, θ, z) by the equation $\rho = e^\theta$ and let the stratum A^1 be the z axis, and $A^2 = S - A^1$.

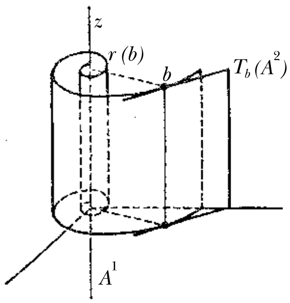


Fig. IV.5.

The strata $A^1 \prec A^2$ do not satisfy Whitney's property B anywhere [the angle between the plane $T_b(A^2)$ and $\overrightarrow{b, r(b)}$ is constant]. Observe that property A is satisfied even though the plane $T_b(A^2)$ has no limit as b tends towards A^1. This is the reason why condition A was not formulated as follows:

A′. $$\boxed{\lim_{b \to a} T_b(B) \supset T_a(A).}$$

Definition. A primary stratification will be called a "*stratification*" if every pair of incident strata $A \prec B$ is *regularly incident* at every point a of A. For example, if we "refine" the primary stratifications of examples 3.7 and 3.8 by adding the origin as a zero-dimensional stratum, we obtain a genuine stratification.

4 Thom's isotopy theorem

4.1

We have seen in §2.6 how the notion of an ambient isotopy is related to the notion of a "locally trivial pair of fibre bundles". Thom's theorem allows one to verify the local triviality of a pair of *differentiable* fibre bundles. Suppose that $\pi : Y \to T$ is a differentiable map from a manifold Y to a *connected* manifold T, and that S is a stratified subset of Y. To lighten the statements, observe that to give a stratified subset S of Y and to give a stratification of the whole of Y, amounts to the same thing, since it is enough to consider the connected components of $Y - S$ as strata also. We say that the *stratified set* Y

is a [locally] *trivial fibre bundle* if there exists a stratified set X and [locally] a homeomorphism of fibre bundles $f : Y \approx X \times T$ which maps every stratum of Y to the product of a stratum of X with the manifold T. This property obviously implies the [local] triviality of the pair (Y, S), and is equivalent to it if the chosen stratification is not too fine compared with the topological structure of S.[9]

Therefore, let $Y \xrightarrow{\pi} T$ be a locally trivial stratified bundle, and assume for a moment that the homeomorphism f which realizes the local trivialization is a diffeomorphism. Then it is clear that every stratum of Y "*submerges*" T, i.e., the restriction of π to each stratum of Y is a map whose *rank is equal to the dimension of* T.[10] Thom's theorem is a kind of converse to this remark, in the case where the projection π is *proper*.[11]

Triviality theorem. Let Y be a stratified set,[12] and let $\pi : Y \to T$ be a *proper* differentiable map from Y onto a connected differentiable manifold T, such that *the restriction of* π *to each stratum of* Y *is of rank equal to the dimension of* T. Then Y is a *locally trivial stratified bundle*.[13]

It is easy to provide counter-examples to the theorem when π is not proper.

4.2 Example (Fig. IV.6). π is the orthogonal projection from the plane $Y = \mathbb{R}^2$ onto the axis $T = \mathbb{R}$, S is the hyperbola $xt = 1$, decomposed into two strata (the two branches of the hyperbola).

The restriction of π to each stratum is of rank 1, but local triviality does not hold at the origin.

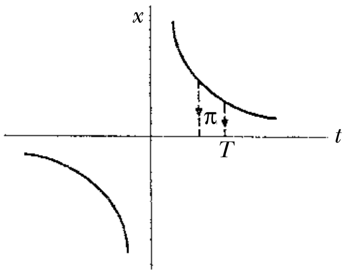


Fig. IV.6.

[9] The exact statement is: if every homeomorphism $f_0 : (X, S_0) \approx (X, S_0)$ leaves each stratum of S_0 globally invariant.

[10] If we assume that T is connected, this map is even surjective.

[11] If $Y = X \times T$, π is proper if and only if X is compact.

[12] To prove this theorem, Thom in fact uses a slightly different definition of the word "stratified", for which the conditions of "regular incidence" are modified a little (cf. Sources, IV).

[13] The theorem says nothing about the *differentiability* of the homeomorphism f which realizes the trivialization. In fact, examples are known where it *cannot* be differentiable.

4.3 Example (Fig. IV.7). Here is a counter-example where the restriction of π is nonetheless surjective:

π is the orthogonal projection of $\mathbb{R}^3 \ni (x, y, t)$ onto the t axis, and S is the surface given by the equation

$$s \equiv x^2 + ty^2 - y = 0, \quad y \leqslant 0$$

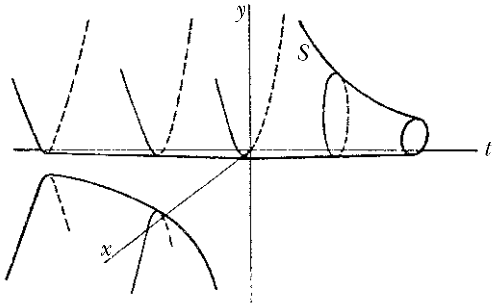


Fig. IV.7.

which is in itself a stratum [S is indeed a manifold, since the differential $ds = 2x\,dx + (2ty - 1)dy + y^2\,dt$ never vanishes on S]. The map $\pi|S$ is clearly surjective, and to see that it is of rank 1, it suffices to notice that on S, the form dt is never proportional to ds. However, local triviality does not hold at the origin (when t crosses the origin, the ellipse S_t becomes a branch of a hyperbola).

4.4 Critical sets. Apparent contours

Let $\pi : Y \to T$ be a differential map from a stratified set Y to a differentiable manifold T. For every stratum A of Y, we define the *critical set of the stratum* A to be the set cA of the points of A where rank $(\pi|A) < \dim T$.[14]

4.5 Lemma. *If* $A \prec B$, $A \cap \overline{cB} \subset cA$.

This lemma can easily be deduced from Whitney's property A. To simplify the argument, we use property A': let a be a point of A, incident to cB. By property A', $T_a(A) \subset \lim_{b \to a} T_b(B)$. Now, if b tends to a along cB, the tangent hyperplane $T_b(B)$ projects onto a hyperplane of dimension $\leqslant q - 1$. The same is therefore true for the hyperplane $\lim_{b \to a} T_b(B)$, and thus for $T_a(A)$.

4.6 Corollary. *The union of the critical sets cB, where B runs over the family of strata of Y, is a closed set.*

[14] Observe the difference between this definition and the usual definition of critical sets: if $\dim A < \dim T$, *the whole manifold A is critical.*

To see this,

$$\overline{cB} = cB \cup \left(\overline{cB} \cap \partial B\right) = cB \cup \bigcup_{A \prec B} \left(\overline{cB} \cap A\right)$$
$$\subset cB \cup \bigcup_{A \prec B} cA \quad \text{by lemma 4.5.}$$

Therefore, since there are a finite number of strata,

$$\overline{\bigcup_B cB} = \bigcup_B \overline{cB} = \bigcup_B cB.$$

Apparent contours

The closed sets $\overline{cA}$ (resp. $\bigcup_A cA$) will be called the *apparent contours at the source* of the stratum A (resp. of the stratified set Y).

Suppose that the projection π is *proper*. Every closed subset of Y projects onto a closed subset of T.[15] The closed sets $\overline{\pi(cA)} = \pi(\overline{cA})$ (resp. $\bigcup_A \pi(cA)$) are called the *apparent contours at the target*, or "*apparent contours*" of the stratum A for short (resp. of the stratified set Y).

By Thom's triviality theorem, we immediately deduce the following proposition:

Let $\pi : Y \to T$ be a *proper* differentiable map from the stratified set Y onto the differentiable manifold T, and let L be the *apparent contour* of Y. Then on every connected component Θ of $T - L$, the stratified set $Y|\Theta$ is a *locally trivial bundle.*

5 Landau varieties

5.1

Let $\pi : Y^p \to T^q$ be a *proper analytic* map between complex analytic manifolds of complex dimensions p and q, respectively. Suppose that the manifold Y is *analytically stratified*, i.e., the nested closed sets $Y = S^{p_0} \supset S^{p_1} \supset S^{p_2} \supset \ldots$ which define its stratification are *complex analytic sets*. The closure of a stratum A is therefore a complex analytic set S_A, which is one of the irreducible components of one of the S_i^p.

5.2 Lemma. *The apparent contour at the source $\overline{cA}$ is a complex analytic set.*

[15] This is an easy exercise of general topology: we simply use the fact that T is locally compact.

Let p_A be the dimension of A, and let

$$\{s_1(y), s_2(y), \ldots, s_r(y)\}$$

be a system of analytic functions which locally span the ideal of S_A (§I.8.3). Let $t_1(y), t_2(y), \ldots, t_q(y)$ be the analytic functions which locally define the projection $\pi : Y^p \to T^q$.

Consider the analytic subset $\Sigma_A \subset S_A$ defined by setting not only the functions s_i to zero, but also all the minors of order $p - p_A + q$ of the matrix

$$\text{(ST)} \qquad \begin{array}{c|c|c|} & i = 1, \ldots, r & j = 1, \ldots, q \\ \hline k = 1, \ldots, p & \dfrac{\partial s_i}{\partial y_k} & \dfrac{\partial t_j}{\partial y_k} \\ \hline \end{array}$$

We claim that $\Sigma_A \cap A = cA$. On the manifold A, the matrix $\|\partial s_i/\partial y_k\|$ is of rank $p - p_A$. Suppose, for the sake of argument, that the $(p - p_A)$-minor of its upper-left corner is non-zero in a neighbourhood of a point $a \in A$. It follows that one can replace $y_1, y_2, \ldots, y_p$ near a by

$$y'_1 = s_1(y), \quad \ldots, \quad y'_{p-p_A} = s_{p-p_A}(y), \quad y'_{p-p_A+1} = y_{p-p_A+1}, \quad \ldots, \quad y'_p = y_p.$$

In these new coordinates, the matrix (ST) can be written

$$\begin{array}{c|c|c|c|} & \multicolumn{2}{c|}{i = 1, \ldots, p - p_A, \ldots, r} & j = 1, \ldots, q \\ \hline k = 1, \ldots, p - p_A & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & ? & ? \\ \hline p - p_A, \ldots, p & 0 & 0 & \dfrac{\partial t_j}{\partial y'_k} \\ \hline \end{array}$$

and the vanishing of all its minors of order $p - p_A + q$ is equivalent to the vanishing of the minors of order q of the matrix

$$\left\| \frac{\partial t_j}{\partial y'_k} \right\|_{k=p-p_A+1,\ldots,p}^{j=1,\ldots,q}$$

which represents the linear tangent map to $\pi|A$.

We therefore have

$$\Sigma_A \cap A = cA, \qquad \text{i.e.,} \qquad cA = \Sigma_A - \Sigma_A \cap \partial A,$$

from which we deduce, using the fact that Σ_A and ∂A are complex analytic sets, that $\overline{cA}$ *is a complex analytic set*, and a component of Σ_A (cf. [1], §44, C).

5.3 Corollary. The apparent contour *(at the target)* $\pi(\overline{cA})$ *is a* complex analytic set. *This follows from Remmert's theorem* ([33]; cf. also [15]) *on the image of a complex analytic set under a proper analytic map.*

Landau varieties

The analytic set $LA = \pi(\overline{cA})$, which is an apparent contour of the stratum A, will be called the *Landau variety*[16] *of the stratum* A. In the sequel, we will only be interested in Landau varieties of codimension 1, and we will therefore assume that

$$\boxed{p_A \geqslant q - 1.}$$

5.4 Singularities of Landau varieties

In order to study Landau varieties, we have at our disposal Thom's results [36] on the singularities of the critical sets of a map.[17] The essential notion, which opens the way to a methodical approach, is that of a "*generic*" map, which is unfortunately too delicate to be made precise here. Let us only note that

1° every map can be approximated by a generic map,
2° every map which is sufficiently close to a generic map is generic.

Here are some general results due to Thom. Generically, we have

$$\boxed{\dim cA = \dim \pi(cA) = q - 1.}$$

The set of points where rank $(\pi|A) = q - 1$ ["*critical points of corank*[18] 1"] generically has *no singularities*. If, furthermore, rank $(\pi|cA) = q - 1$, the critical point is called "*ordinary*" of corank 1. The map π, restricted to the set of ordinary critical points of corank i, is therefore an *immersion* of manifolds.

Here is a particular result, for $\boxed{q = 2}$:[19]

– Generically, all the critical points are of corank 1; the critical curve cA therefore has no singularities; the exceptional critical points (the points where the rank of $\pi|A$ is zero instead of 1) are isolated points, which project onto cusps of the curve $\pi(cA)$.

5.5 Example. Let A be a *torus* embedded in Euclidean space $\mathbb{R}^3$, and let π be the orthogonal projection of this torus onto a plane $\mathbb{R}^2$ (which will be the plane of the diagram). If the angle between this plane and the axis of the torus is sufficiently small, the torus will appear as in Fig. IV.8 or Fig. IV.9.

[16] The use of the word "variety" rather than "manifold" is appropriate since these analytic sets have singularities in general.

[17] Thom was actually interested in the *real differentiable* case, but most of his arguments can be easily transposed to the complex analytic setting (as long as they only concern the *local* structure).

[18] This is the "*corank at the target*", which is equal, by definition, to the dimension of the target minus the rank.

[19] The special cases $q = 3$ and $q = 4$ were studied exhaustively by Thom.

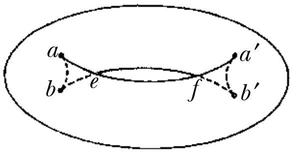


Fig. IV.8.

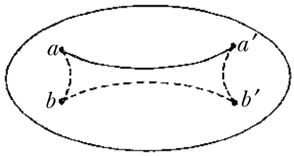


Fig. IV.9.

The critical curve cA has no singularities. What we see in the figure is the projection of this critical curve, which has four cusps a, b, a', b', which are projections of points where the tangent to cA is perpendicular to the plane of the diagram. The generic nature[20] of these cusps can be expressed by the fact that they survive when one changes the angle of the diagram a little. We can observe that Fig. IV.8 also features two double points e and f, but the corresponding branches of the critical curve do not intersect in the source: these points therefore cannot be captured by a *local study of the source.*

5.6 Incidence relations for Landau varieties

A point $u \in LA \cap LB$ will be called an *effective intersection point* of the Landau varieties LA and LB if it is the projection of a point

$$a \in \overline{cA} \cap \overline{cB}.$$

Suppose that $A \prec B$, and that a is an ordinary critical point, of corank 1, for the stratum A. Let $b \in cB$ be an ordinary critical point of corank 1 for the stratum B. When b tends to a, the angle between the tangent planes $T_a(A)$, $T_b(B)$ tends to zero, and so also the angle between their projections. Setting $u = \pi(a)$, $v = \pi(b)$, we therefore find that the tangent $(q-1)$-plane $T_v(LB)$ tends towards the tangent $(q-1)$-plane $T_u(LA)$ as v tends to u. In short, *at an effective intersection point, the Landau varieties of two incident strata are tangent to each other.*

5.7 Example (Fig. IV.10). B is a sphere, A is a great circle of this sphere, and π is the projection onto the plane of the diagram.

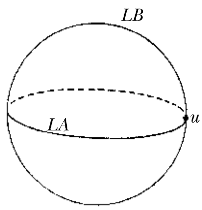


Fig. IV.10.

[20] This *stable* nature of the cusps of Landau curves came as a nasty surprise for physicists, who, having seen it on an example [7], believed their example to be "pathological".

V

Ramification around Landau varieties

1 Overview of the problem

1.1 The fundamental group of a topological space

Recall that a *path* α with initial point u and end point v in a topological space T, is a continuous map $\alpha : [0,1] \to T$ such that $\alpha(0) = u$, $\alpha(1) = v$. If the end point v of a path α coincides with the initial point u' of a path α', we can define a path $\alpha' \cdot \alpha$ with initial point u and end point v' by setting

$$(\alpha' \cdot \alpha)(\tau) = \begin{cases} \alpha(2\tau) & \text{if } 0 \leqslant \tau \leqslant \frac{1}{2}, \\ \alpha'(2\tau - 1) & \text{if } \frac{1}{2} \leqslant \tau \leqslant 1. \end{cases}$$

Intuitively, $\alpha' \cdot \alpha$ is obtained by "travelling" first along the path α, and then along the path α'.

We will denote by α^{-1} the path obtained by "travelling backwards along" α, i.e., the path with initial point v and end point u defined by

$$\alpha^{-1}(\tau) = \alpha(1 - \tau).$$

Two paths α_0 and α_1 with the *same initial point u and the same end point v* are called "*homotopic*" if they can be interpolated by a family of paths α_τ which vary continuously in τ and all have *the initial point u and end point v*.

Homotopy of paths is an equivalence relation, which is compatible with the composition law defined earlier, so one can speak of the composition of two homotopy classes of paths, provided, of course, that the end point of the first coincides with the initial point of the second.

Let us fix once and for all a point u_0 in T, and consider the set of all "*loops based at u_0*", i.e., the set of all paths which have initial point and end point at u_0. The set of *homotopy classes of loops based at u_0*, equipped with the composition law defined above, is a *group*. The identity element is the class

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_5,

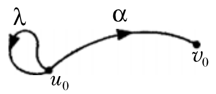
of the constant loop (which maps $[0,1]$ to the point u_0). The inverse of the class of a loop λ is the class of the loop λ^{-1}. The group defined in this way is called the "*fundamental group*, or first homotopy group", *based at* u_0*, of the topological group* T. It is denoted $\pi_1(T, u_0)$.

What happens if we replace the base point u_0 with another one v_0? Let α be a path with base point u_0 and end point v_0.[1] The correspondence

$$[\alpha] : \pi_1(T, u_0) \longrightarrow \pi_1(T, v_0)$$

defined by

$$[\alpha](\lambda) = \alpha \cdot \lambda \cdot \alpha^{-1}$$

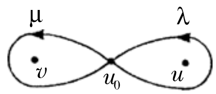


is obviously a *homomorphism* which only depends on the *homotopy class of* α. It is therefore an *isomorphism*,[2] since it has $[\alpha^{-1}]$ as its inverse. This isomorphism can be conveniently interpreted in terms of deformations of loops:

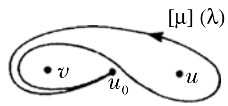
The loop $[\alpha](\lambda) = \alpha \cdot \lambda \cdot \alpha^{-1}$ *can be obtained by a deformation of the loop* λ *during which the base point travels along the path* α.

In particular, by setting $u_0 = v_0$, we obtain in this way a geometric interpretation of the *inner automorphisms* of $\pi_1(T, u_0)$.

Example. Let $T = \mathbb{R}^2 - \{u, v\}$ be the plane minus two points. Then $\pi_1(T, u_0)$ is the free (non-abelian) group spanned by λ and μ:



The element $[\mu](\lambda)$ can be represented by the loop[3]



[1] We will always assume that T is path-connected, so that such a path exists.

[2] Thus, whenever we are only interested in the fundamental group up to isomorphism, there is no need to specify the base point.

[3] In the sequel, no distinction will be made, neither in terminology nor notation, between a "*loop*" and its homotopy class. Thus, from a practical point of view, "drawing a loop", will simply mean to suggest, by drawing an oriented line, the homotopy class of a loop.

1.2 Overview of the central problem of this chapter

Let $Y^p = X^n \times T^q$ $(p = n + q)$ be a product of complex analytic manifolds, where the manifold X is assumed to be *compact*, and let $\pi : Y \to T$ be the natural projection map.[4] Let S be an analytically stratified subset of Y, and let L be the Landau variety given by the apparent contour of S in T. We saw in the previous chapter that $(Y, S)|T - L$ is therefore a locally trivial pair of bundles, and thus every homotopy class $\lambda \in \pi_1(T - L, u_0)$ defines an ambient isotopy class for S_{u_0} in the fibre Y_{u_0}, and therefore an automorphism

$$\lambda_* : H_*(Y_{u_0}, S_{u_0}) \longrightarrow H_*(Y_{u_0}, S_{u_0})$$

or

$$H_*(Y_{u_0} - S_{u_0}) \longrightarrow H_*(Y_{u_0} - S_{u_0}).$$

The correspondence $\lambda \rightsquigarrow \lambda_*$ defines a homomorphism $\star$ from the fundamental group of the base space to the group of automorphisms of H_*:

$$\star : \pi_1(T - L, u_0) \longrightarrow \operatorname{Aut} H_*(Y_{u_0}, S_{u_0}),$$

and the study of this homomorphism is the main subject of this chapter.

In order to describe it precisely, we will need to make the following two hypotheses:

Hypothesis 1.[5] S is a union of closed submanifolds S_1, $S_2, \ldots,$ of complex codimension 1, and in general position (with the obvious stratification).

A stratum A of S will be of the form

$$A = S_1 \cap S_2 \cap \cdots \cap S_m - \bigcup_{k>m} S_k,$$

and every point $a \in A$ will have a neighbourhood V which does not meet S_k for $k > m$, and is equipped with coordinates $y_1, y_2, \ldots, y_p$ such that $S_i \cap V$ is the hyperplane $y_i = 0$ $(i = 1, 2, \ldots, m)$, where the point a is at the origin of these coordinates. We will set $\pi(V) = W$, and equip W with the local coordinates t_1, $t_2, \ldots, t_q$ having the point $u = \pi(a)$ as the origin. The projection π will therefore be given in V by a system of analytic functions $t_1(y), t_2(y), \ldots, t_q(y)$. Observe that we have preferred to choose coordinates where the S_i can be written simply, at the cost of obscuring the product structure of Y.

Hypothesis 2. T is simply connected, i.e., $\pi_1(T) = 0$.

[4] The hypothesis (cf. the previous chapter) that Y is a product is superfluous and is only to simplify the exposition.

[5] The reason for this hypothesis is the impossibility of studying all imaginable stratified structures! We point out, nonetheless, that a recent theorem allows one to reduce any analytic set to a set satisfying (Hyp. 1), due to a procedure of "blowing-up singularities" [17].

We will call a "*simple loop*" based at u_0, a loop λ constructed in the following way: we are given

1° a point $u \in L$, which is smooth of complex codimension 1;
2° a path θ in $T - L$, with initial point u_0 and end point u_1 near u;
3° in a neighbourhood of u, we can choose coordinates $t_1, t_2, \dots, t_q$ such that L is given by the equation $t_1 = 0$. We thus define a "*small loop*" ω based at u_1 by making the complex variable t_1 trace a small circle around the origin in the positive direction, whilst the other variables $t_2, \dots, t_q$ stay fixed.

The "simple loop" λ is therefore defined by

$$\lambda = \left[\theta^{-1}\right](\omega) = \theta^{-1} \cdot \omega \cdot \theta.$$

1.3 Proposition. $\pi_1(T) = 0 \Leftrightarrow \pi_1(T - L)$ *is spanned by simple loops.*

($\Leftarrow$) is obvious, since every "small loop", and therefore every simple loop, is homotopic to zero in T.

To prove ($\Rightarrow$), we must show that in $T - L$, every loop λ with base point u_0 is homotopic to a sequence of simple loops. Now, by hypothesis, λ is homotopic to zero in T, i.e., there exists a continuous map $\Lambda : \square \to T$ from the square to the space T, which coincides with λ on the lower side of the square, and maps the three other sides to the point u_0. By a classical theorem in homotopy theory, the continuous map Λ can be approximated by a differentiable map Λ', which can itself be approximated (by a "*transversality theorem*") by a map Λ'' which is *transversal*[6] *on* L. $\Lambda''^{-1}(L)$ therefore consists of isolated points, which are finite in number since $\Lambda''^{-1}(L)$ is closed in the compact set $\square$, and these points $\tau_1, \tau_2, \dots, \tau_k$ are the inverse images of points of L, which are smooth of real codimension 2.

It therefore suffices to replace the lower side of the square, considered as a path in $\square$, by a sequence of simple loops as indicated by Fig. V.1 and to send everything into $T - L$ via the map Λ''.

[6] The notion of a map which is *transversal on a stratified set* is defined in technical note (10). Here we exploit the fact, proved by Whitney [42], that every complex analytic set can be stratified; its strata of maximal dimension therefore obviously consist of smooth points.

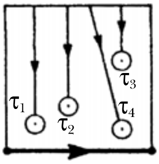


Fig. V.1.

2 Simple pinching. Picard–Lefschetz formulae

2.1 Description of a simple pinching

With the notations of section 1 (Hyp. 1), let us suppose that a is an *ordinary critical point on the stratum A, of corank* 1 *at the target, and not in the closure of the critical sets of the other strata.*

We then know that near a, the critical set cA is generically without singularities, and projects isomorphically onto LA. We can therefore choose local coordinates such that

$$LA \cap W = \{t_1 = 0\},$$

$$cA \cap V = \{y_1 = y_2 = \cdots = y_{n+1} = 0\} \subset A \cap V = \{y_1 = y_2 = \cdots = y_m = 0\}$$

and that

$$t_2(y) = y_{n+2}, \quad \ldots, \quad t_q(y) = y_p$$

(the fact that the map $\pi | cA \cap V : cA \cap V \to LA \cap W$ is an isomorphism means that the Jacobian determinant of $t_2, \ldots, t_q$ with respect to $y_{n+2}, \ldots, y_p$ is non-zero).

It only remains to determine the function $t_1(y)$. When restricted to A, it will generically have (cf. Thom [36]) a quadratic singularity on cA and we can ensure that it can be written

$$t_1(0, \ldots, 0, y_{m+1}, \ldots, y_p) = y_{m+1}^2 + \cdots + y_{n+1}^2.$$

Furthermore, since a is not critical for any of the manifolds in the star of A, each of the terms $y_1, \ldots, y_m$ will necessarily appear linearly in the function $t_1(y)$, which can therefore be written

$$t_1(y) = y_1 + y_2 + \cdots + y_m + y_{m+1}^2 + \cdots + y_{n+1}^2.$$

What does this mean for the fibre bundle structure of Y? If we set

$$x_1 = y_2, \quad x_2 = y_3, \quad \ldots, \quad x_n = y_{n+1},$$

we see immediately that $(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_q)$ is a system of coordinates in the neighbourhood V, in which the projection π can simply be written

$(x,t) \overset{\pi}{\rightsquigarrow} (t)$. In this new system, the equations of the manifolds S_1, $S_2, \dots, S_m$ are

$$\begin{aligned} s_1(x,t) &\equiv t_1 - (x_1 + x_2 + \cdots + x_{m-1} + x_m^2 + \cdots + x_n^2) = 0, \\ s_2(x,t) &\equiv x_1 = 0, \\ s_3(x,t) &\equiv x_2 = 0, \\ &\vdots \\ s_m(x,t) &\equiv x_{m-1} = 0. \end{aligned}$$

In the fibre $Y_t \approx X$, the manifolds $S_{it} = Y_t \cap S_i$ are in general position, except along $t_1 = 0$, where we will say that they have a "*simple pinching*" or a "*quadratic pinching*". More precisely, the expression "quadratic pinching" will be reserved for the case $m \leqslant n$, where squares genuinely occur in $s_1(x,t)$, and the case $m = n+1$ will be called "*linear pinching*":

$$\begin{aligned} s_1(x,t) &\equiv t_1 - (x_1 + x_2 + \cdots + x_n) = 0, \\ s_2(x,t) &\equiv x_1 = 0, \\ &\vdots \\ s_{n+1}(x,t) &\equiv x_n = 0. \end{aligned}$$

[This is the case where the Landau variety LA is simply the *projection* of the stratum A, of dimension $q-1$.[7]]

Fig. V.2 represents the situation near a simple pinching, in small dimensions. We work inside the open set $U_t = T_t \cap V$, equipped with the local coordinates $(x_1, x_2, \dots, x_n)$ as before, and the parameter t_1 has been chosen to be real positive, and sufficiently small so that all the interesting part of the figure is contained in the open set U_t.

2.2 Description of the vanishing chains

In Fig. V.2, we have also depicted some chains which will play an important role. All these chains "vanish" when $t_1 \to 0$, hence their name. We catalogue them below.

The "*vanishing cell*" $\boldsymbol{e}$ is the real cell bounded by the manifolds $S_1, S_2, \dots, S_m$:[8]

$$\boldsymbol{e} \begin{cases} x_1, x_2, \dots, x_n \text{ real}, \\ s_1, s_2, \dots, s_m \geqslant 0. \end{cases}$$

Its orientation is chosen arbitrarily: in the figures, we have chosen the orientation determined by the system of real axes $(\operatorname{Re} Ox_1, \operatorname{Re} Ox_2, \dots, \operatorname{Re} Ox_n)$.

[7] Since we always suppose that near u, $\operatorname{codim} LA = 1$, the cases where $m > n+1$ are excluded.

[8] In what follows, we will no longer write the index t of the manifolds S_{it} or of the open sets U_t: it is understood that we will stay in the fibre Y_t, with $t = (t_1, 0, \dots, 0)$, t_1 real positive and small.

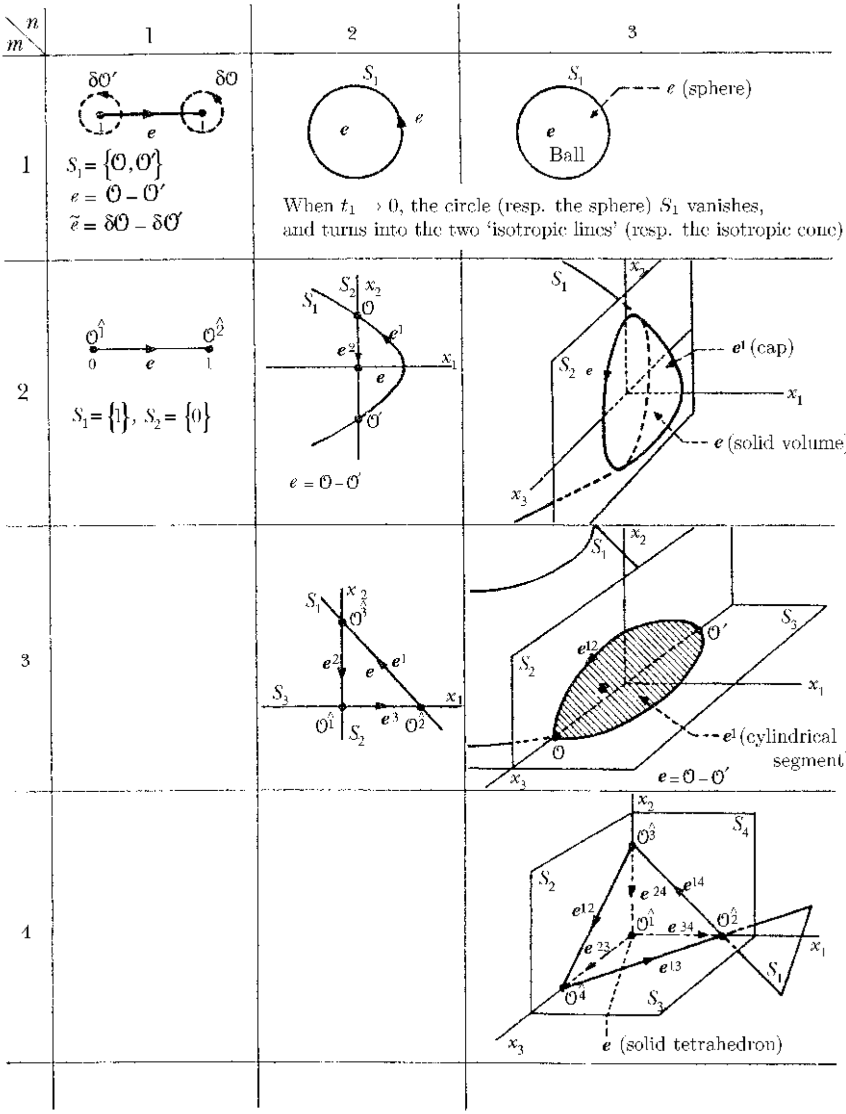


Fig. V.2.

The "*vanishing sphere*" e is the iterated boundary of the vanishing cell: $e = \partial_m \circ \partial_{m-1} \circ \cdots \circ \partial_1 \boldsymbol{e}$, where ∂_i is the operation which consists in taking the part of the boundary which lies in the manifold S_i.

$$e \begin{cases} x_1, x_2, \ldots, x_n \text{ real}, \\ s_1 = s_2 = \cdots = s_m = 0. \end{cases}$$

The "*vanishing cycle*" $\widetilde{e}$ is the iterated coboundary of the vanishing sphere:

$$\widetilde{e} = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_m e,$$

where δ_i denotes Leray's coboundary "with respect to" the manifold S_i.

More generally, we define for every increasing sequence of integers

$$\{i_1 < i_2 < \cdots < i_\mu\} \subset \{1, 2, \ldots, m\},$$

$$\boldsymbol{e}^{i_1 i_2 \ldots i_\mu} = \partial_{i_\mu} \circ \cdots \circ \partial_{i_2} \circ \partial_{i_1} \boldsymbol{e}$$

and

$$\widetilde{e}_{j_1 j_2 \ldots j_{m-\mu}} = \delta_{i_1} \circ \delta_{i_2} \circ \cdots \circ \delta_{i_\mu} \boldsymbol{e}^{i_1 i_2 \ldots i_\mu},$$

where $\{j_1, j_2, \ldots, j_{m-\mu}\}$ is the increasing sequence which is complementary to $\{i_1, i_2, \ldots, i_\mu\}$. The way to remember this notation is the following: the upper indices indicate the manifolds S_i which contain the chain being considered; the lower indices indicate the manifolds S_i which bound the chain being considered; a bold typeface indicates that the chain being considered is bounded by *all* the manifolds S_i which do not contain it; the upper tilde indicates that the chain being considered meets *none* of the varieties S_i, except those which bound it.

The case $m = n + 1$ (*linear pinching*) is somewhat special. The vanishing sphere e, and as a consequence, the vanishing cycle $\widetilde{e}$ do not exist. The chains

$$\boldsymbol{e}^{1.2\ldots\widehat{\imath}\ldots n+1} = \partial_{n+1} \circ \cdots \circ \widehat{\partial_i} \circ \cdots \circ \partial_1 \boldsymbol{e}$$

reduce (up to a sign), to points which will be denoted

$$\mathscr{O}^{\widehat{\imath}} = \text{the point } \bigcap_{j \neq i} S_j.$$

2.3 Characterization of the vanishing classes

The previous description was in a particular system of axes. It is good to have more intrinsic criteria for recognizing vanishing chains, or more precisely, the homology classes which they represent. Let

$$e(U \cap S_1 \cap S_2 \cap \cdots \cap S_m) \in H_{n-m}(U \cap S_1 \cap S_2 \cap \cdots \cap S_m)$$

be the homology class of the vanishing sphere, let

$$\boldsymbol{e}(U, S_1 \cup S_2 \cup \cdots \cup S_m) \in H_n(U, S_1 \cup S_2 \cup \cdots \cup S_m)$$

be the relative homology class of the vanishing cell, and let

$$\widetilde{e}(U - S_1 \cup S_2 \cup \cdots \cup S_m) \in H_n(U - S_1 \cup S_2 \cup \cdots \cup S_m)$$

be the homology class of the vanishing cycle, etc.

The manifold $U \cap S_1 \cap \cdots \cap S_m$ is a complex $(n-m)$-sphere, which deformation retracts onto a "real" $(n-m)$-sphere[9] and its homology group $H_{n-m}(U \cap S_1 \cap \cdots \cap S_m)$ is the infinite cyclic group generated by the class of this sphere. This is the characterization of the vanishing sphere. On the other hand, one can show that in the open set U, all the boundary maps ∂_i and coboundaries δ_i are *isomorphisms* between suitable homology groups. All these homology groups are therefore infinite cyclic[10] and spanned by the corresponding vanishing classes, which are thus characterized without ambiguity with respect to the vanishing sphere. In practice, that means that, for example, if we had known how to construct a compact chain in U whose m-fold iterated boundary was the vanishing sphere e, this chain would be [homologous to] the vanishing cell $\boldsymbol{e}$.

2.4 Localization. Picard's formula

It follows from the hypotheses of section 1 that, in order to solve the problem of the ramification of homology, in other words, to know the action of the fundamental group $\pi_1(T-L)$ on the various homology groups of the fibre Y_t, it is enough to know the action of a "*small loop*" ω, defined in a neighbourhood of a smooth point $u \in L$, of complex codimension 1. As we saw in §2.1, this point u is *in general* the projection of an *ordinary critical point* $a \in S_1 \cap S_2 \cap \cdots \cap S_m$ which corresponds to a *simple pinching* of the manifolds $S_{1u}, S_{2u}, \ldots, S_{mu}$. It is intuitively quite clear that, in these conditions, the problem can be *localized* in the fibre of the ball U which surrounds the pinching point a. More precisely, one shows that if h is a homology class of the fibre, represented by a cycle Γ, the transformed class $\omega_* h$ can be represented by a cycle Γ' *which only differs from* Γ *in the interior of the ball* U.

The correspondence $\Gamma \rightsquigarrow \Gamma' - \Gamma$ thus defines a homomorphism, which we will denote by "Var" (for "*variation*"), from the homology of X to the

---

[9] For an open set U (which we shall call a *ball*) which is large enough to contain the "real" $(n-m)$-sphere.

[10] Except in the following two *marginal cases*, where the homology in dimension zero modifies the situation a little:
The case $m = n$: $H_n(U - S_1 \cup S_2 \cup \cdots \cup S_n)$ is the free group on two generators

$$\delta_1 \circ \cdots \circ \delta_n \mathscr{O} \qquad \text{and} \qquad \delta_1 \circ \cdots \circ \delta_n \mathscr{O}',$$

where $\mathscr{O}'$ and $\mathscr{O}' \in H_0(U \cap S_1 \cap S_2 \cap \cdots \cap S_n)$ are the classes of the two points which make up the 0-sphere $U \cap S_1 \cap S_2 \cap \cdots \cap S_n$. Observe that

$$e = \mathscr{O} - \mathscr{O}' \qquad \text{and} \qquad \widetilde{e} = \delta_1 \circ \cdots \circ \delta_n \mathscr{O} - \delta_1 \circ \cdots \circ \delta_n \mathscr{O}'.$$

The case $m = n+1$: $H_n(U - S_1 \cup S_2 \cup \cdots \cup S_{n+1})$ is the free group on $n+1$ generators

$$\delta_1 \circ \cdots \circ \widehat{\delta_i} \circ \cdots \circ \delta_{n+1} \mathscr{O}^{\widehat{\imath}} \quad (i = 1, 2, \ldots, n+1).$$

homology of U, for example:

$$\begin{aligned} \mathrm{Var}: H_n(X-S) &\longrightarrow H_n(U-S), \\ \mathrm{Var}: H_n(X,S) &\longrightarrow H_n(U,S) \end{aligned}$$

or, more generally,

$$\mathrm{Var}: H_n(X - S_{\mu+1\ldots m}, S_{1.2\ldots\mu}) \longrightarrow H_n(U - S_{\mu+1\ldots m}, S_{1.2\ldots\mu}).$$

From now on we will use the shorthand notation

$$S_{i_1 i_2 \ldots i_\mu} = \bigcup_{i=i_1,i_2,\ldots,i_\mu} S_i, \qquad S^{i_1 i_2 \ldots i_\mu} = \bigcap_{i=i_1,i_2,\ldots,i_\mu} S_i.$$

Now we pointed out in §2.3 that each of the homology groups of U is *cyclic*, generated by the appropriate *vanishing chain*. As a result, the variation of a homology class

$$\begin{aligned} & h \in H_{n-m}(S_1 \cap S_2 \cap \cdots \cap S_m), \\ \text{resp.}\quad & \widetilde{h} \in H_n(X-S) \\ \text{resp.}\quad & \boldsymbol{h} \in H_n(X,S), \\ \text{resp.}\quad & \widetilde{h}_{1.2\ldots\mu} \in H_n(X - S_{\mu+1\ldots m}, S_{1.2\ldots\mu}), \end{aligned}$$

will necessarily be of the form

$$\left.\begin{array}{lll} (P) & \mathrm{Var}\, h & = N\, e \\ \left(\widetilde{P}\right) & \mathrm{Var}\, \widetilde{h} & = N\, \widetilde{e} \\ (\boldsymbol{P}) & \mathrm{Var}\, \boldsymbol{h} & = N\, \boldsymbol{e} \\ \left(\widetilde{P}_{1.2\ldots\mu}\right) & \mathrm{Var}\, \widetilde{h}_{1.2\ldots\mu} & = N\, \widetilde{e}_{1.2\ldots\mu} \end{array}\right\} \text{(“Picard's formulae”)},$$

where N is an integer to be determined, which depends on h (resp. on $\widetilde{h}$, $\boldsymbol{h}$, $\widetilde{h}_{1.2\ldots\mu}$).

2.5 Lefschetz' formula

This formula determines the integer N which occurs in Picard's formula. The formula which corresponds to formula $(\widetilde{P})$ is

$$(\widetilde{L}) \qquad \boxed{N = (-)^{(n+1)(n+2)/2} \langle \boldsymbol{e} \mid \widetilde{h} \rangle.}$$

To understand this intuitively, note that the *intersection index* $\langle \boldsymbol{e} \mid \widetilde{h} \rangle$ represents the number of times that $\widetilde{h}$ travels along the vanishing cell $\boldsymbol{e}$, which is the cell bounded by the manifolds $S_1, S_2, \ldots, S_m$. It can therefore be thought of as the “*linking number*” of h with the manifolds $S_1, S_2, \ldots, S_m$.

From formula $\widetilde{L}$, one can easily deduce the formulae (L), $(\boldsymbol{L})$, $(\widetilde{L}_{1.2\ldots\mu})$:

$$(L) \qquad \boxed{N = (-)^{(n-m+1)(n-m+2)/2}\langle e \mid h\rangle.}$$

To prove (L), we set $\widetilde{h} = \delta_1 \circ \cdots \circ \delta_m h$. The coboundary operation naturally commutes with Var, so that

$$\operatorname{Var} \widetilde{h} = \delta_1 \circ \cdots \circ \delta_m \ \operatorname{Var} h = \delta_1 \circ \cdots \circ \delta_m N\, e = N\, \widetilde{e};$$

but by $(\widetilde{L})$,

$$N = (-)^{(n+1)(n+2)/2}\langle \boldsymbol{e} \mid \widetilde{h}\rangle,$$

from which we deduce (L) by transposing ∂ and δ

$$(\boldsymbol{L}) \qquad \boxed{N = (-)^{(n-m+1)(n-m+2)/2}\langle e \mid h\rangle, \quad \text{where} \quad h = \partial_m \circ \cdots \circ \partial_1 \boldsymbol{h}.}$$

The boundary operation naturally commutes with Var, giving

$$\operatorname{Var} h = \partial_m \circ \cdots \circ \partial_1 \ \operatorname{Var} \boldsymbol{h} = \partial_m \circ \cdots \circ \partial_1 N\, \boldsymbol{e} = N\, e,$$

and it only remains to apply (L).

$$(\widetilde{L}_{1.2\ldots\mu}) \qquad \boxed{\begin{array}{c} N = (-)^{(n-\mu+1)(n-\mu+2)/2}\left\langle e^{1.2\ldots\mu} \mid \widetilde{h}^{1.2\ldots\mu}\right\rangle, \\ \text{where} \quad \widetilde{h}^{1.2\ldots\mu} = \partial_\mu \circ \cdots \circ \partial_1 \widetilde{h}_{1.2\ldots\mu} \end{array}}$$

which is left as an exercise for the reader.[11]

2.6 Ramification type

Let us begin by establishing the *intersection formulae for the classes of vanishing cycles*: by §II.7.4,

$$\langle e \mid e\rangle = \begin{cases} 2(-)^{(n-m)(n-m+1)/2} & \text{if } n-m+1 \text{ is odd,} \\ 0 & \text{if } n-m+1 \text{ is even.} \end{cases}$$

From this we deduce, by transposition (cf. §III.2.3),

$$\left\langle \boldsymbol{e}^{1.2\ldots\mu} \mid \widetilde{e}^{1.2\ldots\mu}\right\rangle = \begin{cases} 2(-)^{(n-\mu)(n-\mu+1)/2} & \text{if } n-m+1 \text{ is odd,} \\ 0 & \text{if } n-m+1 \text{ is even,} \end{cases}$$

[11] We observe that the Picard–Lefschetz formulae hold true even in the marginal cases of §2.3, where the argument in §2.4 is no longer correct. In the case $m = n+1$, where the vanishing sphere e does not exist, one must make the convention that the corresponding "vanishing class" is *zero*, and thus the variation of h or $\widetilde{h}$ is *still zero* (because e and $\widetilde{e} = 0$) and likewise for the variation of $\boldsymbol{h}$ (because $N = \langle e \mid h\rangle = 0$).

where
$$\widetilde{e}^{1.2\ldots\mu} = \delta_{\mu+1} \circ \cdots \circ \delta_m e = \partial_\mu \circ \cdots \circ \partial_1 \widetilde{e}_{1.2\ldots\mu}.$$
The Picard–Lefschetz formula therefore gives

$$\omega_* \widetilde{e}_{1.2\ldots\mu} = \widetilde{e}_{1.2\ldots\mu} + (-)^{(n-\mu+1)(n-\mu+2)/2} \langle \boldsymbol{e}^{1.2\ldots\mu} \mid \widetilde{e}_{1.2\ldots\mu} \rangle \widetilde{e}_{1.2\ldots}$$
$$= \begin{cases} -\widetilde{e}_{1.2\ldots\mu} & \text{if } n-m+1 \text{ is odd,} \\ \ \ \widetilde{e}_{1.2\ldots\mu} & \text{if } n-m+1 \text{ is even.} \end{cases}$$

These formulae can be easily established directly.

We deduce that:

If $n-m+1$ *is odd*, two circuits around L brings $\widetilde{h}_{1.2\ldots\mu}$ back to its original branch:
$$\omega_*^2 \widetilde{h}_{1.2\ldots\mu} = \omega_* (\widetilde{h}_{1.2\ldots\mu} + N \, \widetilde{e}_{1.2\ldots\mu}) = \widetilde{h}_{1.2\ldots\mu};$$
which signifies *ramification of “square root” type.*

If $n-m+1$ *is even*, every new circuit around L adds the same multiple of the vanishing class to $\widetilde{h}_{1.2\ldots\mu}$:

$$\omega_*^k \widetilde{h}_{1.2\ldots\mu} = \widetilde{h}_{1.2\ldots\mu} + k \, N \, \widetilde{e}_{1.2\ldots\mu},$$

which signifies *ramification of “logarithmic” type.*

2.7 Invariant classes

An *invariant class* will be the name given to a homology class which contains a cycle situated *outside the open set* U. By §2.4, this property *implies* the absence of ramification, i.e., $\omega_* h = h$, but is *a priori* stronger.

2.8 Proposition. *The class* $\widetilde{h}_{1.2\ldots\mu}$ *is invariant if and only if the intersection indices of* $\widetilde{h}_{1.2\ldots\mu}$ *with all the elements of* $H_n(U - S_{1.2\ldots\mu}, S_{\mu+1\ldots m})$ *are zero.*

We will admit this proposition without proof, which can be proved using Poincaré duality (§II.5.5).

2.9 Corollary.[12] *The class* $\widetilde{h}_{1.2\ldots\mu}$ *is invariant if and only if the index* N *of formula* $\widetilde{L}_{1.2\ldots\mu}$ *is zero.*

This is because $H_n(U - S_{1.2\ldots\mu}, S_{\mu+1\ldots m})$ is spanned[12], by §2.3, by the vanishing cycle $\widetilde{e}_{\mu+1\ldots m}$, whose intersection with $\widetilde{h}_{1.2\ldots\mu}$ gives precisely the index N in Lefschetz' formula (after transposing ∂ and δ).

[12] This is not true in the marginal cases ($\mu = m = n$ and $\mu = m = n+1$).

2.10 Lemma. *Let $\widetilde{h}_{1.2\ldots\mu}$ be an arbitrary class. Then*

$$\widetilde{h}^{1.2\ldots\mu} = \partial_\mu\circ\cdots\circ\partial_1\widetilde{h}_{1.2\ldots\mu} \quad \textit{and} \quad N = (-)^{(n-\mu+1)(n-\mu+2)/2}\langle \boldsymbol{e}^{1.2\ldots\mu} \mid \widetilde{h}^{1.2\ldots\mu}\rangle.$$

If $n-m+1$ is odd, the class

$$\widetilde{h}'_{1.2\ldots\mu} = 2\widetilde{h}^{1.2\ldots\mu} + N\,\widetilde{e}_{1.2\ldots\mu}$$

is invariant.

This lemma is an immediate consequence of corollary 2.9 and the intersection formula for vanishing cycles (§2.6).

2.11 Proof of the Picard–Lefschetz formula, when $n-m+1$ is odd[13]

By lemma 2.10,

$$\omega_*\widetilde{h}'_{1.2\ldots\mu} = \widetilde{h}'_{1.2\ldots\mu}.$$

Given that $\omega_*\widetilde{e}_{1.2\ldots\mu} = -\widetilde{e}_{1.2\ldots\mu}$, we deduce that

$$2\omega_*\widetilde{h}_{1.2\ldots\mu} - N\,\widetilde{e}_{1.2\ldots\mu} = 2\widetilde{h}_{1.2\ldots\mu} + N\,\widetilde{e}_{1.2\ldots\mu},$$

in other words,

$$2\omega_*\widetilde{h}_{1.2\ldots\mu} = 2\widetilde{h}_{1.2\ldots\mu} + 2N\,\widetilde{e}_{1.2\ldots\mu},$$

which, after dividing by 2, gives the Picard–Lefschetz formula.[14]

2.12 Proof of the Picard–Lefschetz formula, when $\mu = 0$, $m = n+1$

We must prove that if $m = n+1$, $\omega_*\widetilde{h} = \widetilde{h}$. Let

$$N = (-)^{(n+1)(n+2)/2}\left\langle \boldsymbol{e} \mid \widetilde{h}\right\rangle.$$

By duality,

$$\left\langle \boldsymbol{e} \mid \delta_1\circ\delta_2\circ\cdots\circ\delta_n\widehat{\mathscr{O}^{n+1}}\right\rangle = (-)^{n(n+1)/2}\left\langle \widehat{\mathscr{O}^{n+1}} \mid \widehat{\mathscr{O}^{n+1}}\right\rangle = (-)^{n(n+1)/2},$$

therefore the class

$$\widetilde{h}' = \widetilde{h} + (-)^n N\,\delta_1\circ\delta_2\circ\cdots\circ\delta_n\widehat{\mathscr{O}^{n+1}}$$

is invariant (cf. corollary 2.9).

---

[13] The proof in the even case will not be given here.

[14] This division by 2 is legitimate, because we know from (§2.4) that the variation of $\widetilde{h}_{1.2\ldots\mu}$ is an element in the group $H_n(U - S_{\mu+1\ldots m}, S_{1.2\ldots\mu})$, which is a *free* group.

Now, obviously,

$$\omega_* \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n \widehat{\mathscr{O}^{n+1}} = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n \omega_* \widehat{\mathscr{O}^{n+1}}$$
$$= \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n \widehat{\mathscr{O}^{n+1}},$$

so that $\omega_* \widetilde{h} = \widetilde{h}$.

We observe that if $N \neq 0$, *the class* $\widetilde{h}$ *is not invariant in the sense of* §2.7, *even though it is not ramified.*

3 Study of certain singular points of Landau varieties

The Picard–Lefschetz formulae allow one *in principle* to solve the ramification problem completely. The recipe goes as follows:

(i) Calculate the fundamental group $\pi_1(T - L, u_0)$, and choose a family of "*simple loops*" (§2.1) which generate it.
(ii) Calculate the homology groups of the fibre Y_{u_0}, and identify which elements are "*vanishing cycles*" corresponding to each simple loop: a homology class h will be "vanishing" for a simple loop $\lambda = \theta^{-1} \cdot \omega \cdot \theta$ if $\theta_* h$ is one of the vanishing cycles defined in §2.3.
(iii) Calculate all the intersection indices for all the homology classes which arise in the problem.
(iv) All the ingredients are ready: it only remains to apply the Picard–Lefschetz formulae.

Apart from some simple cases, this recipe is obviously very indigestible! In this section, we will restrict ourselves to studying two local models, which will not only be of illustrative value but they will also solve the ramification problem in the neighbourhood of a *generic* singular point on Landau varieties. The two types of singular points we will study are (cf. chap. IV, §5):

1° the effective intersection points of two Landau varieties,
2° the cusps on a Landau curve.

We remark in passing that the four stages of the "recipe" proposed above are not independent, and this fact is worth studying in greater detail. In particular, knowing the fundamental group $\pi_1(T - L, u_0)$ already provides information about the vanishing classes and their intersections.

First model: effective intersection of two Landau varieties

3.1

We consider two incident strata:

$$A = S_1 \cap S_2 \cap \cdots \cap S_m \cap S_{m+1} - \bigcup_{j>m+1} S_j,$$

$$B = S_1 \cap S_2 \cap \cdots \cap S_m - \bigcup_{j>m} S_j$$

and we work inside the open set $V \subset Y$ ($V \cap S_j = \varnothing$ for $j > m+1$), equipped with the coordinates $(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_q)$ such that

$$\begin{aligned} s_1(x,t) &\equiv t_1 - \left[x_1 + x_2 + \cdots + x_{m-1} + (x_m - t_2)^2 + x_{m+1}^2 + \cdots + x_n^2\right], \\ s_2(x,t) &\equiv x_1, \\ &\;\vdots \\ s_m(x,t) &\equiv x_{m-1}, \\ s_{m+1}(x,t) &\equiv x_m. \end{aligned}$$

This is the *generic* situation described in chapter IV, §5.6, expressed in particular coordinates.

Example: $m = 1$, $n = 2$:

$$\begin{aligned} s_1 &\equiv t_1 - \left[(x_1 - t_2)^2 + x_2^2\right] \quad \text{(Fig. V.3)}, \\ s_2 &\equiv x_2. \end{aligned}$$

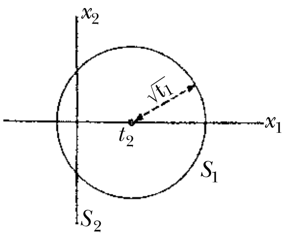


Fig. V.3.

The critical sets are given by

$$\begin{aligned} cA &= \left\{(x,t) : x_1 = x_2 = \cdots = x_n = 0, t_1 - t_2^2 = 0\right\}, \\ cB &= \{(x,t) : \text{all the } x_i \text{ are zero except } x_m,\ x_m = t_2, t_1 = 0\} \end{aligned}$$

and the Landau varieties are therefore

$$LA : t_1 - t_2^2 = 0, \qquad LB : t_1 = 0 \quad \text{(Fig. V.4)}.$$

We will set $L = LA \cup LB$, and we will suppose, to simplify the notation, that the chart $(t_1, t_2, \ldots, t_q)$ on the open ball $W = \pi(V)$ sends this open set onto the whole of $\mathbb{C}^q$.

Since the only coordinates which appear explicitly are t_1, t_2, there is no loss of generality in assuming that $q = 2$.

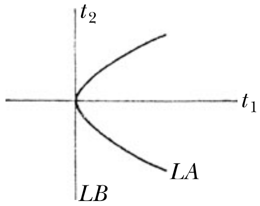


Fig. V.4.

3.2 Calculation of $\pi_1(W-L)$

The projection pr: $(t_1, t_2) \rightsquigarrow t_1$ makes $W-L$ a fibre bundle with base $\mathbb{C}-\{0\}$ and fibre

$$\mathrm{pr}^{-1}(t_1) = \mathbb{C} - \left\{\sqrt{t_1}\right\} \cup \left\{-\sqrt{t_1}\right\} \approx \mathbb{C} - \{1\} \cup \{-1\}.$$

In the fibre $\mathrm{pr}^{-1}(1)$, consider the loops α, α' based at $u_0 = (1,0)$ represented in Fig. V.5, and let γ be the loop based at u_0 defined by

$$\gamma : \tau \rightsquigarrow (t_1 = e^{2i\pi\tau}, t_2 = 0) \quad \text{(Fig. V.6)}.$$

Fig. V.5.
The complex t_2-plane (*fibre*).

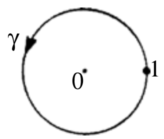


Fig. V.6.
The complex t_1-plane (*base*).

α, α', γ *are loops of* $W-L$, *satisfying the relations*

$$\gamma \cdot \alpha \cdot \gamma^{-1} = \alpha',$$
$$\gamma \cdot \alpha' \cdot \gamma^{-1} = \alpha,$$

(which are obvious from the geometric interpretation of conjugation: cf. §1.1).

It is now a simple technical affair[15] to show that α, α', γ *generate the group* $\pi_1(W-L, u_0)$ *and do not satisfy any other relations than the ones indicated*: in other words, $\pi_1(W-L, u_0)$ is the quotient of the (non-abelian) free group with generators α, α', γ by the subgroup spanned by $\gamma\cdot\alpha\cdot\gamma^{-1}$ and $\gamma\cdot\alpha'\cdot\gamma^{-1}$.[16]

[15] The technique used is the "*exact sequence for the homotopy of a fibre bundle*".

[16] More simply, it is the quotient of the free group with generators α, γ by the subgroup spanned by $\gamma^2 \cdot \alpha \cdot \gamma^{-2} \cdot \alpha^{-1}$.

3.3 Choice of simple loops

Clearly, α and α' are simple loops, but not γ (because the origin, which γ winds around, is not a smooth point of L). On the other hand, the loops $\beta = \alpha^{-1} \cdot \gamma$ et $\beta' = \alpha'^{-1} \cdot \gamma$ are simple, as the schematic Fig. V.7 suggests.

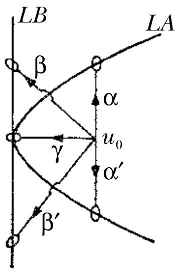


Fig. V.7.

(The interpretation of this picture is left to the imagination of the reader.)

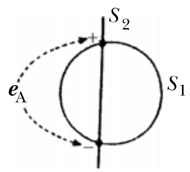
Now, some obvious algebraic operations show that $\pi_1(W - L, u_0)$ *is spanned by the simple loops* α, α', β, β', *related by*

$$(\alpha\beta) \qquad (\alpha \cdot \beta) = \alpha' \cdot \beta' = \beta \cdot \alpha' = \beta' \cdot \alpha.$$

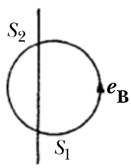
3.4 Description of some vanishing chains (t_1, t_2 real; $t_1, t_1 - t_2^2 > 0$)

Spheres:

$$e_A \begin{cases} x_1, \ldots, x_n \text{ real}, \\ s_1 = \cdots = s_m = s_{m+1} = 0, \end{cases}$$

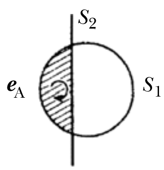


$$e_B \begin{cases} x_1, \ldots, x_n \text{ real}, \\ s_1 = \cdots = s_m = 0. \end{cases}$$

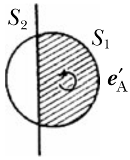


Cells:

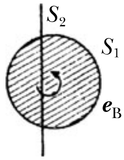
$$\boldsymbol{e}_A \begin{cases} x_1, \ldots, x_n \text{ real}, \\ s_1, \ldots, s_m > 0, \quad s_{m+1} < 0, \end{cases}$$



$$\boldsymbol{e}'_A \begin{cases} x_1, \dots, x_n \text{ real}, \\ s_1, \dots, s_m > 0, \quad s_{m+1} > 0, \end{cases}$$



$$\boldsymbol{e}_B \begin{cases} x_1, \dots, x_n \ \text{ real}, \\ s_1, \dots, s_m > 0. \end{cases}$$



We see that the cell $\boldsymbol{e}_B$ is the union of the two cells $\boldsymbol{e}_A$ and $\boldsymbol{e}'_A$.

If we have arbitrarily chosen an orientation for one of the chains above, the orientation of all the others will in fact be determined by the relations:

(A) $\qquad e_A = \partial_{m+1} \circ \partial_m \circ \cdots \circ \partial_1 \boldsymbol{e}_A = \partial_{m+1} \circ \partial_m \circ \cdots \circ \partial_1 \boldsymbol{e}'_A,$

(B) $\qquad e_B = \partial_m \circ \cdots \circ \partial_1 \boldsymbol{e}_B,$

(AB) $\qquad \boldsymbol{e}_B = \boldsymbol{e}'_A - \boldsymbol{e}_A.$

Observe that formula (AB) must contain $\boldsymbol{e}_A$ and $\boldsymbol{e}'_A$ with *opposite* signs, otherwise (A) and (B) would not be compatible with the fact that e_B is a cycle: $\partial_{m+1} e_B = 0$.

3.5 Homologies

Since $U \cap S_1 \cap \cdots \cap S_m \cap S_{m+1}$ is a complex $(n-m-1)$-sphere, its homology group in dimension $n-m-1$ is infinite cyclic, generated by the class of e_A. Likewise, $U \cap S_1 \cap \cdots \cap S_m$ is a complex $(n-m)$-sphere, whose homology group in dimension $n-m$ is generated by e_B.

We easily deduce[17] that $H_n(U, S_1 \cup \cdots \cup S_m \cup S_{m+1})$ is the free abelian group with generators $\boldsymbol{e}_A$ and $\boldsymbol{e}_B$.

In the same way,[18] we show that $H_n(U - S_1 \cup \cdots \cup S_m \cup S_{m+1})$ is a free abelian group with two generators $\widetilde{e}_A$ and $\widetilde{e}_B$, and hence

$$\widetilde{e}_A = \delta_1 \circ \cdots \circ \delta_m \circ \delta_{m+1} e_A,$$

whereas $\widetilde{e}_B$ is a cycle of

$$U - S_1 \cup \cdots \cup S_m \cup S_{m+1},$$

which is homologous in $U - S_1 \cup \cdots \cup S_m$ to $\delta_1 \circ \cdots \circ \delta_m e_B$. Note that this prescription only defines the class $\widetilde{e}_B$ up to a multiple of $\widetilde{e}_A$. Geometrically, this ambiguity represents the various ways in which the cycle $\delta_1 \circ \cdots \circ \delta_m e_B$ can be moved away from the manifold S_{m+1}.

[17] Using "exact homology sequences".

[18] This uses a special case of the "decomposition theorem" [12] which can be deduced from "Leray's exact homology sequence".

3.6 Application of the Picard–Lefschetz formula[19]

The action of a *simple loop* λ ($\lambda = \alpha,\ \alpha',\ \beta,\ \beta'$) on a class $\widetilde{h} \in H_n(X-S)$ is given by the Picard–Lefschetz formula:

$$\lambda_* \widetilde{h} = \widetilde{h} + N_\lambda \widetilde{e}_\lambda; \quad N_\lambda = (-)^{(n+1)(n+2)/2} \left\langle \boldsymbol{e}_\lambda \mid \widetilde{h} \right\rangle,$$

where $\widetilde{e}_\lambda$ is a class of $H_n(X-S)$, which comes from the injection of a class from $H_n(U-S)$, whereas the intersection index N_λ only depends on the class of the vanishing cell $\boldsymbol{e}_\lambda$ in $H_n(U,S)$.

It is easy to determine which of the cells of §3.4 vanish for a given loop λ: in this way, we find that

$$\boldsymbol{e}_\alpha = \boldsymbol{e}_A, \quad \boldsymbol{e}_{\alpha'} = \boldsymbol{e}'_A, \quad \boldsymbol{e}_\beta = \boldsymbol{e}_{\beta'} = \boldsymbol{e}_B.$$

Relation (AB) thus gives

$$\text{(N)} \qquad \boxed{N_\beta = N_{\beta'} = N_{\alpha'} - N_\alpha.}$$

On the other hand, the sphere e_A (resp. e_B) vanishes for α, α' (resp. β, β') giving

$$\widetilde{e}_\alpha = \widetilde{e}_{\alpha'} = \widetilde{e}_A,$$

and $\widetilde{e}_\beta$ (and $\widetilde{e}_{\beta'}$) $= \widetilde{e}_B$, up to a multiple of $\widetilde{e}_A$.

We can always choose $\widetilde{e}_B = \widetilde{e}_\beta$, and therefore $\widetilde{e}_{\beta'} = \widetilde{e}_B + \nu \widetilde{e}_A$ where ν is an integer which will be determined later.

Applying the Picard–Lefschetz formula twice, we get

$$\begin{aligned}
\beta_* \alpha'_* \widetilde{h} &= \alpha'_* \widetilde{h} + (-)^{(n+1)(n+2)/2} \left\langle \boldsymbol{e}_B \mid \alpha'_* \widetilde{h} \right\rangle \widetilde{e}_B \\
&= \widetilde{h} + (-)^{(n+1)(n+2)/2} \left[\left\langle \boldsymbol{e}'_A \mid \widetilde{h} \right\rangle \widetilde{e}_A + \left\langle \boldsymbol{e}_B \mid \widetilde{h} + N_{\alpha'} \widetilde{e}_A \right\rangle \widetilde{e}_B \right] \\
&= \widetilde{h} + N_{\alpha'} \widetilde{e}_A + N_\beta \widetilde{e}_B + (-)^{(n+1)(n+2)/2} N_{\alpha'} \left\langle \boldsymbol{e}_B \mid \widetilde{e}_A \right\rangle \widetilde{e}_B.
\end{aligned}$$

Then, by transposition,

$$\langle \boldsymbol{e}_B \,|\, \widetilde{e}_A \rangle = \langle 0 \,|\, e_A \rangle = 0;$$

$$\begin{tikzcd}
\boldsymbol{e}_B \arrow[d, "\partial_1"'] & \widetilde{e}_A \\
\vdots \arrow[d, "\partial_m"'] & \vdots \arrow[u, "\delta_1"'] \\
e_B \arrow[d, "\partial_{m+1}"'] & {} \arrow[u, "\delta_m"'] \\
0 & e_A \arrow[u, "\delta_{m+1}"']
\end{tikzcd}$$

[19] Here we will only treat the ramification of the homology group $H_n(X-S)$, leaving the other homology groups to the reader.

and therefore
$$\beta_* \alpha'_* \widetilde{h} = \widetilde{h} + N_{\alpha'} \widetilde{e}_A + N_\beta \widetilde{e}_B.$$
Likewise,
$$\begin{aligned}\beta'_* \alpha_* \widetilde{h} &= \widetilde{h} + N_\alpha \widetilde{e}_A + N_{\beta'}(\widetilde{e}_B + \nu \widetilde{e}_A)\\ &= \widetilde{h} + (N_\alpha + \nu N_{\beta'})\widetilde{e}_A + N_{\beta'} \widetilde{e}_B.\end{aligned}$$

Setting these two expressions equal, as $(\alpha\beta)$ would suggest, and using (N), we find that
$$\boxed{\nu = 1.}$$

Exercise. By continuing to exploit the relations $(\alpha\beta)$, show that
$$\langle \boldsymbol{e}_A \mid \widetilde{e}_B \rangle = (-)^{n(n+1)/2}$$
and that
$$\langle \boldsymbol{e}'_A \mid \widetilde{e}_B \rangle = \Big\{ \mp(-)^{n(n+1)/2} \quad \text{for } n-m \begin{cases}\text{even}\\ \text{odd}\end{cases}$$
(to obtain this last relation, one can use the intersection formulae from §2.6).

Second model: cusps on a Landau curve

3.7

Let A^k be a complex analytic manifold[20] of dimension $k > 2$, and let $\pi : A^k \to T$ be an analytic map from A^k to a complex analytic manifold of dimension two. The generic situation called a cusp can be represented by the local model:
$$\begin{aligned}\pi :&(\xi_1, \xi_2, \dots, \xi_k) \rightsquigarrow (t_1, t_2),\\ &t_1 = \xi_1,\\ &t_2 = \xi_2^3 - \xi_1\xi_2 + \xi_3^2 + \xi_4^2 + \dots + \xi_k^2.\end{aligned}$$
The critical set is the "parabola"
$$cA^k = \big\{\xi : 3\xi_2^2 - \xi_1 = 0, \xi_3 = \xi_4 = \dots = \xi_k = 0\big\}$$
which projects onto the Landau curve given parametrically by
$$L : t_1 = 3\xi_2^2, \quad t_2 = -2\xi_2^3,$$
i.e.,
$$L = \{t : 4t_1^3 - 27t_2^2 = 0\} \quad \text{(Fig. V.8)}.$$

[20] Considering this manifold A^k as a *stratum* in a bigger manifold, as we have done until now, essentially adds nothing.

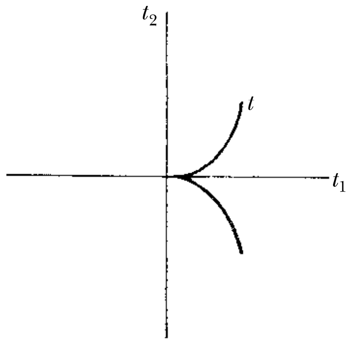


Fig. V.8.

3.8 Calculation of the fundamental group $\pi_1(W - L, u_0)$

Let us choose the point $u_0 = (1, 0)$ as base point and let λ, μ be simple loops associated to each of the two real branches of the Landau curve (Fig. V.10).

Proposition. *$\pi_1(W - L, u_0)$ is spanned by λ and μ, subject to the relation*

$$(\lambda\mu) \qquad \boxed{\lambda \cdot \mu \cdot \lambda = \mu \cdot \lambda \cdot \mu.}$$

An amusing way to find this relation is to use the manifold A^2 depicted in Fig. V.9.

$A^2 - \pi^{-1}(L)$ is a "triple cover"[21] of $W - L$.

Now the equation of $\pi^{-1}(L)$ is

$$4t_1^3(\xi) - 27t_2^2(\xi) \equiv 4\xi_1^3 - 27\left(\xi_2^3 - \xi_1\xi_2\right)^2 \equiv \left(3\xi_2^2 - \xi_1\right)^2\left(4\xi_1 - 3\xi_2^2\right) = 0,$$

in other words $\pi^{-1}(L)$ is the union of two parabolas which are tangent to each other (one of which is obviously cA^2).

The fundamental group $\pi_1(A^2 - \pi^{-1}(L),\, a)$ [where $a = (\xi_1 = 1,\ \xi_2 = 0)$] is therefore, by §3.3, generated by the simple loops α, α', β, β' (Fig. V.11) subject to the relations

$$\alpha \cdot \beta = \alpha' \cdot \beta' = \beta \cdot \alpha' = \beta' \cdot \alpha.$$

The projection π induces a homomorphism of fundamental groups:

$$\pi_* : \pi_1(A^2 - \pi^{-1}(L), a) \longrightarrow \pi_1(W - L, u_0);$$
$$\pi_*\alpha = \lambda^2,$$
$$\pi_*\alpha' = \mu^2,$$
$$\pi_*\beta = \lambda^{-1}\mu\lambda,$$
$$\pi_*\beta' = \mu^{-1}\lambda\mu.$$

[21] I.e., a bundle, with discrete fibres consisting of three points.

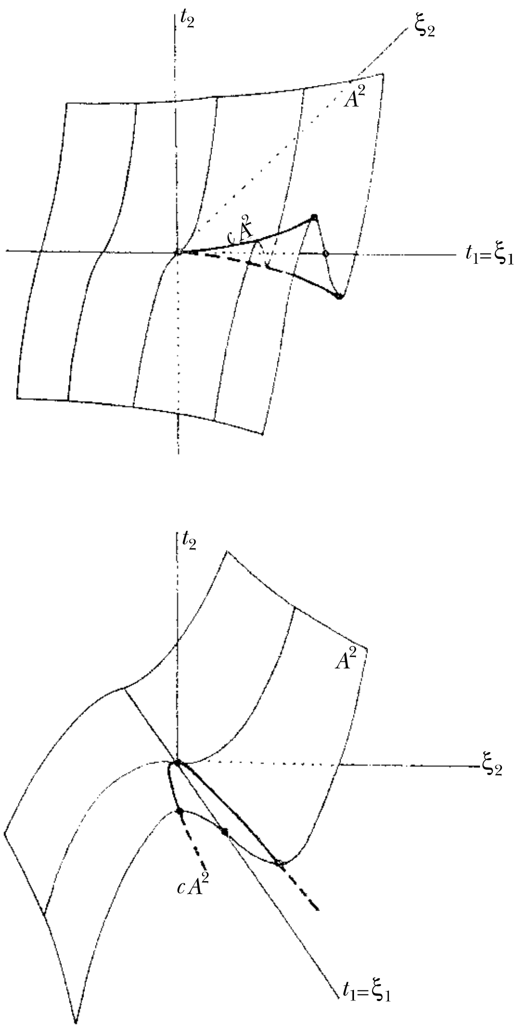


Fig. V.9. Graph of the map $\pi : A^2 \to T$ ("Whitney's fold") [41].

The first two relations are obvious, if we recall that the loops α, α' wind around quadratic critical points of the projection π. As for the last two, the reader, with a bit of imagination, can read them off Fig. V.12.

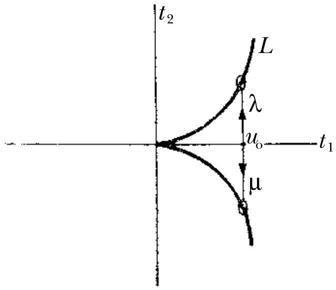


Fig. V.10.

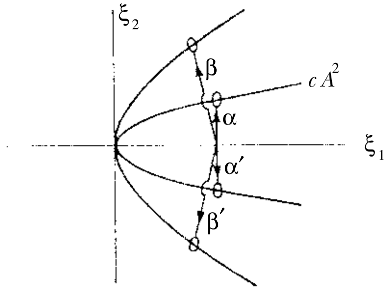


Fig. V.11.

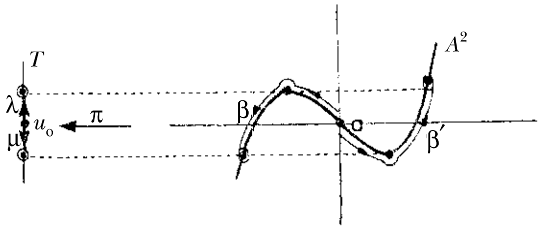


Fig. V.12.

Given relations $(\alpha\beta)$, we obtain $(\lambda\mu)$. It is once again a simple technical matter[22] to show that λ and μ generate $\pi_1(W-L, u_0)$ and are subject to no relations other than $(\lambda\mu)$.

3.9 Homology of the fibre

Let $S_t^{k-2} = \pi^{-1}(t)$ be the "fibre" of the manifold A^k. It is easy to see[23] that, for $t \notin L$, the group $H_{k-2}(S_t^{k-2})$ is free abelian on two generators. At the point $u_0 = (1, 0)$, where

$$S_{u_0}^{k-2} = \left\{(\xi_2, \xi_3, \ldots, \xi_k) : s(\xi) \equiv \xi_2(\xi_2 - 1)(\xi_2 + 1) + \xi_3^2 + \cdots + \xi_k^2 = 0\right\},$$

---

[22] Again using the exact homotopy sequence for bundles.

[23] If we place ourselves at the point $t = (0, t_2)$,

$$S_t^{k-2} = \{(\xi_2, \xi_3, \ldots, \xi_k) : \xi_2^3 + \xi_3^2 + \cdots + \xi_k^2 = t_2\}$$

which is a special case of the situation studied in [29].

these two generators can be represented by the "spheres" (Fig. V.13)

$$e_\lambda \begin{cases} \xi_2 \text{ real}, -1 \leqslant \xi_2 \leqslant 0, \\ \xi_3, \dots, \xi_k \text{ pure imaginary}, \\ s(\xi) = 0; \end{cases}$$

$$e_\mu \begin{cases} \xi_2 \text{ real}, 0 \leqslant \xi_2 \leqslant 1, \\ \xi_3, \dots, \xi_k \text{ real}, \\ s(\xi) = 0. \end{cases}$$

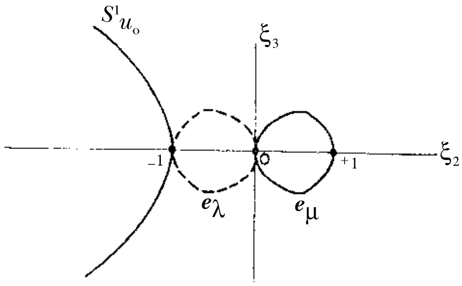


Fig. V.13.

These "spheres" are vanishing spheres with respect to the loops λ and μ, respectively. They intersect transversally in a single point (the origin), and if we orient them by the system of indicators $(\operatorname{Im} O\,\xi_3, \dots, \operatorname{Im} O\,\xi_k)$ and $(\operatorname{Re} O\,\xi_2, \dots, \operatorname{Re} O\,\xi_\lambda)$ respectively, their intersection index is

$$\langle e_\mu \mid e_\lambda \rangle = (-)^{(k-1)(k-2)/2}.$$

3.10 Application of the Picard–Lefschetz formula

The ramification of a class $h \in H_{k-2}(S^k)$ is given by the Picard–Lefschetz formulae

$$\lambda_* h = h + N_\lambda e_\lambda, \qquad N_\lambda = (-)^{k(k-1)/2} \langle e_\lambda \mid h \rangle,$$
$$\mu_* h = h + N_\mu e_\mu, \qquad N_\mu = (-)^{k(k-1)/2} \langle e_\mu \mid h \rangle.$$

Exercise. Using the Picard–Lefschetz formulae and the relation $\lambda\mu\lambda = \mu\lambda\mu$, rederive the intersection formula $\langle e_\mu \mid e_\lambda \rangle = \mp 1$. With the choice of orientations of §3.9, show that

$$(\lambda\mu\lambda)_* h = (\mu\lambda\mu)_* h = h + (N_\lambda - N_\mu)\, e_\lambda + \left(N_\mu - (-)^k N_\lambda\right) e_\mu$$

and

$$\left(\lambda\mu\lambda^{-}1\right)_* h = h + (N_\lambda - N_\mu)\,(e_\lambda - e_\mu)\,.$$

VI

Analyticity of an integral depending on a parameter

In this chapter, we will study integrals of the following type:

$$J(t) = \int_\Gamma \varphi_t, \tag{0}$$

where Γ is a (compact) chain in a complex analytic manifold X, and φ_t is a differential form on X (which may have singularities) *and which depends holomorphically on the parameter* t (where $t \in T$, a complex analytic manifold). In section 1 we will show that, under certain conditions, the function $J(t)$ remains holomorphic whenever it is possible to deform the chain Γ continuously. There will be certain submanifolds S of X which will play a special role for this deformation: those submanifolds which will have to *bound* the chain Γ, and those submanifolds which the chain Γ will have to "*avoid*" (the singular loci of the form φ_t). In this way, the existence of an *ambient isotopy* (chap. IV, §1) of S inside X, which will obviously guarantee the existence of such a deformation, will ensure at the same time that the function $J(t)$ is holomorphic. It can therefore only have singularities along the *Landau varieties* defined in chapter IV (§5).

In section 2, we will study in detail the nature of the singularity along such a Landau variety L.

1 Holomorphy of an integral depending on a parameter

We begin with two obvious lemmas, for which the manifold X only needs to be differentiable, and not necessarily complex analytic.

1.1 Lemma. *Hypothesis: The form φ_t, which depends holomorphically on t, remains regular in a neighbourhood of the support of Γ, as t varies in a neighbourhood of $t_0 \in T$.*

Conclusion: $J(t)$ is holomorphic at the point t_0.

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_6,

Proof. By differentiating under the integral, we see that expression (0) is infinitely differentiable with respect to the coordinates (t), and that its differential with respect to the complex conjugate coordinates $(\bar{t})$ vanishes. $\square$

1.2 Lemma. *Now let us allow (continuous) deformations of the chain Γ in t, but let us strengthen the hypotheses of lemma 1.1 in the following way:*

1° $\Gamma(t)$ is a cycle,

2° φ_t is regular and closed in a neighbourhood $U(t)$ of the support of $\Gamma(t)$, $\forall\, t \in W$, an open subset of T.

Then $J(t)$ is holomorphic in W.

Proof. Since φ_t is closed, the integral only depends on the homology class $h_*(U(t))$ of $\Gamma(t)$.

Now, since $\Gamma(t)$ varies continuously, it is clear that for t' sufficiently close to t, $\Gamma(t)$ is a cycle of $U(t')$ which is homologous to $\Gamma(t')$

$$\Gamma(t) \in h_*(U(t')).$$

Therefore $J(t') = \int_{\Gamma(t)} \varphi_{t'}$, and lemma 1.1 shows that the function $J(t')$ is holomorphic at the point t. $\square$

1.3 Remark. In lemma 1.2, we could obviously have assumed that $\Gamma(t)$ is a *relative* cycle modulo a submanifold S, by adding the hypothesis $\varphi_t|S = 0$.

1.4

From now on, we will assume that the manifold X is *complex analytic*. Let $S_t \subset X$ be a *closed* complex analytic submanifold of codimension 1, which depends analytically on t (i.e., its local equations are analytic functions of t).

Let φ_t be a differential form which depends holomorphically on t, which is regular and *closed* in $X - S_t$, and let ω_t be a form on S_t which belongs to the *residue class* of φ_t

$$\omega_t \in \mathrm{Res}[\varphi_t].$$

Therefore, if $\gamma(t)$ is a cycle in S_t which is deformed continuously with t, the function

$$J(t) = \int_{\gamma(t)} \omega_t$$

is holomorphic.

This is because the residue theorem states that

$$J(t) = \int_{\delta\gamma(t)} \varphi_t,$$

and we are in the situation of lemma 1.2, with

$$U(t) = X - S_t.$$

1.5

We have already treated the case where $\Gamma(t)$ is a *relative* cycle of X, modulo a *fixed* submanifold (in remark 1.3). Now let us treat the case where this submanifold varies analytically with t.

Let $S_t \subset X$ be a closed, complex analytic submanifold of codimension 1, which depends analytically on t, let $\Gamma(t)$ be a *relative cycle* of (X, S_t), which is deformed continuously in t, and let φ_t be a *closed* regular differential form on X, such that $\varphi_t | S_t = 0$ (for example, this will be the case for *every holomorphic form of maximal degree*).

Suppose, furthermore, that:

(φ_1) φ_t can locally be written in a neighbourhood of every point $y \in S_t$:

$$\varphi_t(x) = [s_y(x,t)]^\alpha \, \omega_y(x),$$

where $\omega_y(x)$ is a differential form which does not depend on t and defined near y, α is an integer $\geqslant 0$, and $s_y(x,t)$ is a local equation of S_t near y, which is assumed to satisfy the conditions[1]:

(φ_2) $\dfrac{\partial s_y(x,t)/\partial t}{s_y(x,t)}$ is independent of y,

(φ_3) $\dfrac{d s_y(x,t)}{s_y(x,t)} \wedge \omega_y(x)$ is independent of t (where d is the differential for constant t).

Conditions (φ_2) and (φ_3) are consequences of (φ_1), for $\alpha > 0$. In fact, (φ_2) follows from the equation

$$\frac{\partial \varphi_t}{\partial t} = \alpha \frac{\partial s_y/\partial t}{s_y}\, \varphi_t,$$

whereas (φ_3) can be deduced from

$$0 = d\varphi_t(x) = [s_y(x,t)]^\alpha \left[\alpha \, \frac{d s_y(x,t)}{s_y(x,t)} \wedge \omega_y(x) + d\omega_y(x) \right].$$

1.6 Proposition. *With the hypotheses of §1.5, the function*

$$J(t) = \int_{\Gamma(t)} \varphi_t$$

is holomorphic.

---

[1] Observe that (φ_2) is satisfied if $s(x,t)$ is a *global* equation and (φ_3) is satisfied if ω is a *holomorphic form of maximal degree.*

Proof. The integral only depends on the relative homology class $h_*(X, S_t)$ of $\Gamma(t)$. Let $\gamma(t) = \partial\Gamma(t)$.

We construct, as in §III.2.2, a tubular neighbourhood V of S_{t_0} and a retraction $\mu : V \to S_{t_0}$.

For $y \in S_{t_0}$, and t near t_0, S_t cuts the disk $\mu^{-1}(y)$ in only one point $x(y,t)$. As y travels along the cycle $\gamma(t_0)$, the geodesic $\overline{yx}_t$ which joins y and $x(y,t)$, sweeps out a chain Γ_{t_0t} whose boundary in S_{t_0} is $-\gamma(t_0)$. Clearly, $\Gamma(t_0) + \Gamma_{t_0t} \in h_*(X, S_t)$, and therefore

$$J(t) = \int_{\Gamma(t_0)} \varphi_t + \int_{\Gamma_{t_0t}} \varphi_t.$$

The first integral is holomorphic by lemma 1.1. In order to study the second, observe that the function $\log(s_y(t)/s_y(t_0))$, which does not depend on y by (φ_2), is a single-valued function on the disk $\mu^{-1}(y)$ minus the cut $\overline{yx}_t$. Its discontinuity along the cut is $2\pi i$, and therefore

$$\int_{\Gamma_{t_0t}} \varphi_t = \frac{1}{2\pi i}\int_{\Delta_{t_0t}} \left(\log\frac{s_y(t)}{s_y(t_0)}\right)\varphi_t,$$

where the cycle Δ_{t_0t} is obtained from the chain Γ_{t_0t} by replacing the cut $\overline{yx}_t$ by a circuit which runs along both edges of the cut (Fig. VI.1). Since the integrand $\left(\log\frac{s_y(t)}{s_y(t_0)}\right)\varphi_t$ is a *closed* differential form [to see this, use (φ_3)], we can replace the cycle Δ_{t_0t} by the cycle $\delta_\mu\gamma(t_0)$ which is homologous to it, and lemma 1.1 shows that the integral obtained in this way is holomorphic. □

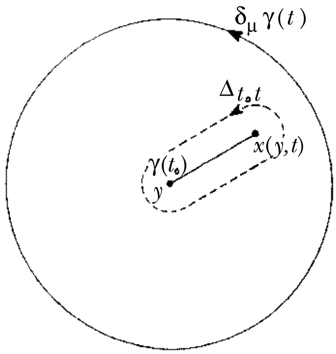


Fig. VI.1.

1.7

In the sequel, instead of considering a *single* submanifold S, we will consider a *family* of complex analytic submanifolds of codimension one in general position $\{S_{1t}, S_{2t}, \dots, S_{mt}\}$, which depend analytically on t.

We then show, exactly as in §1.4, that if φ_t is a differential form which depends holomorphically on t, which is regular and *closed* on $X - (S_1 \cup S_2 \cup \cdots \cup S_m)_t$, whose *iterated residue* is the form ω_t on $(S_1 \cap S_2 \cap \cdots \cap S_m)_t$:

$$\omega_t \in \mathrm{Res}^m[\varphi_t],$$

and if $\gamma(t)$ is a cycle of $(S_1 \cap S_2 \cap \cdots \cap S_m)_t$, which is deformed continuously as a function of t, *then the function*

$$J(t) = \int_{\gamma(t)} \omega_t$$

is holomorphic.

Likewise, proposition 1.6 has the following generalization:

1.8 Proposition. *Let $\Gamma(t)$ be a relative cycle of $(X, (S_1 \cup S_2 \cup \cdots \cup S_m)_t)$, which is deformed continuously as a function of t, and let φ_t be a regular closed differential form on X, such that $\varphi_t | S_{it} = 0$ ($i = 1, 2, \ldots, m$).*

Suppose, furthermore, that

(φ_1) *φ_t can be written locally in the neighbourhood of every point*

$$y \in (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_\mu})_t \quad (\{i_1, i_2, \ldots, i_\mu\} \subset \{1, 2, \ldots, m\}),$$
$$\varphi_t(x) = \prod_{i = i_1, i_2, \ldots, i_\mu} [s_{iy}(x,t)]^{\alpha_i} \omega_y(x),$$

where $\omega_y(\alpha)$ is a differential form which is independent of t, defined near y, $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ are integers $\geqslant 0$, and $s_{iy}(x,t)$ is the local equation of S_{it} near y, which is assumed to satisfy the conditions:

(φ_2) $\dfrac{\partial s_{iy}/\partial t}{s_{iy}}$ *is independent of y;*

(φ_3) $\dfrac{\partial s_{iy}}{s_{iy}} \wedge \omega_y$ *is independent of t.*

Then the function $J(t) = \int_{\Gamma(t)} \varphi_t$ is holomorphic.

Proof. Suppose first of all that

$$\begin{array}{ccccccccc} T & = & T^{(1)} & \times & T^{(2)} & \times & \ldots & \times & T^{(m)} \\ \Cup & & \Cup & & \Cup & & & & \Cup \\ t & = & (t^{(1)}, & & t^{(2)}, & & \ldots, & & t^{(m)}) \end{array}$$

and that the function $s_{iy}(x,t)$ only depends on the variable $t^{(i)}$ ($i = 1, 2, \ldots, m$). Then we prove, exactly as in proposition 1.8, that $J(t)$ is holomorphic in every variable $t^{(1)}, t^{(2)}, \ldots, t^{(m)}$ separately, and therefore (by Hartogs' theorem) also in t.

In the general case, we reduce to the preceding case by enlarging the space of parameters, i.e., by embedding T in $\underbrace{T \times T \times \cdots \times T}_{m\ \text{times}}$ via the diagonal map

$$t \rightsquigarrow (t, t, \ldots, t). \qquad \square$$

2 The singular part of an integral which depends on a parameter

The analysis in chapter V (§2) will enable us to evaluate, at a smooth point $u \in L$ which corresponds to a simple pinching, the singular part of an integral over a cycle

$$\widetilde{\Gamma}_{1.2\ldots\mu} \in \widetilde{h}_{1.2\ldots\mu} \in \mathit{\Pi}_n(X - S_{\mu+1\ldots m}, S_{1.2\ldots\mu})$$

(we repeat the notations of chapter V, §2).

The integral we will study can be written

$$J_\alpha(t) = \int_{\widetilde{h}_{1.2\ldots\mu}} \left(\prod_{i=1}^{\mu} \frac{|-s_j(x,t)|^{-\alpha_j}}{(-\alpha_j)!}\right) \frac{\omega}{\prod_{k=\mu+1}^{m}[s_k(x,t)]^{\alpha_k}}, \tag{2.0}$$

where $\alpha=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ is a family of integers satisfying $\alpha_1,\alpha_2,\ldots,\alpha_\mu \leqslant 0$ and $\alpha_{\mu+1},\alpha_{\mu+2},\ldots,\alpha_m > 0$. The same letter α will also denote the integer $\alpha = \sum_{i=1}^m \alpha_i$. Here, ω is a differential form on X, which is *holomorphic of degree* n ($n = \dim X$), which is independent of t, and the $s_i(x,t)(i = 1,2,\ldots,m)$ are the equations of the manifolds S_i. To simplify the notations, these equations will be assumed to be global: more generally, we could assume that the expressions $s_i(x,t)$, as well as the differential form ω, depend on the chart being considered (although the whole integrand will remain independent of it), but the same precautions as in section 1 will be required to ensure the analyticity of $J_\alpha(t)$ outside the Landau variety (cf. proposition 1.8).

2.1

Let $l(t)$ be the local equation of the Landau variety L near the point $u \in L$. *The behaviour of the function $J_\alpha(t)$ near the point u is given by the formulae*
(2.1′) $n+m-1$ odd:

$$J_\alpha(t) = -\frac{N}{2}\frac{(2\pi i)^{m-\mu}A\prod_{i=1}^m(-\lambda_i)^{\alpha_i}}{(\alpha_{\mu+1}-1)!\cdots(\alpha_m-1)!}\frac{[l(t)]^{\frac{n+m-1}{2}-\alpha}}{\Gamma\left(1+\frac{n+m-1}{2}-\alpha\right)}(1+\mathrm{o}(t))+\text{hol. fn.};$$

(2.1″) $n+m-1$ even $\geqslant 2\alpha$:

$$J_\alpha(t) = N\frac{(2\pi i)^{m-\mu-1}A\prod_{i=1}^m(-\lambda_i)^{\alpha_i}}{(\alpha_{\mu+1}-1)!\cdots(\alpha_m-1)!} \times \frac{[l(t)]^{\frac{n+m-1}{2}-\alpha}}{\left(\frac{n+m-1}{2}-\alpha\right)!}\log l(t)(1+\mathrm{o}(t)) + \text{hol. fn.};$$

(2.1‴) $n+m-1$ even $< 2\alpha$:

$$J_\alpha(t) = -N\frac{(2\pi i)^{m-\mu-1}A\prod_{i=1}^m(-\lambda_i)^{\alpha_i}}{(\alpha_{\mu+1}-1)!\cdots(\alpha_m-1)!}\cdot\frac{\left(\alpha-\frac{n+m-1}{2}-1\right)!}{[-l(t)]^{\alpha-\frac{n+m-1}{2}}}(1+\mathrm{o}(t)) + N\log l(t)\times\text{hol. fn.} + \text{hol. fn.},$$

and is completed by the following two equations:

(2.1$^{\mathrm{IV}}$) $\mu = m = n+1$: $J_\alpha(t)$ is holomorphic;
(2.1$^{\mathrm{V}}$) $m = n+1$, $\mu = 0$: $J_\alpha(t)$ has a polar singularity

$$J_\alpha(t) = (-)^{n+1} N \frac{(2\pi i)^n A \prod_{i=1}^{n+1} \lambda_i^{\alpha_i}}{(\alpha_i - 1)! \cdots (\alpha_{n+1} - 1)!} \frac{(\alpha - n - 1)!}{[l(t)]^{\alpha - n}} (1 + \mathrm{o}(t)) + \text{hol. fn.}$$

In these formulae, hol. fn. means a function which is holomorphic at the point u, and $\mathrm{o}(t)$ denotes a function which is holomorphic and vanishes at the point u. The number N is the intersection index given by Lefschetz' formula. The λ_i are the coefficients of the expansion of the differential form $\pi^* dl$ (i.e., dl viewed as a form on Y) at the pinching point a:

$$\pi^* dl = \sum_{i=1}^{m} \lambda_i ds_i \tag{2.2}$$

(the existence of this expansion comes from the fact that $\pi^* dl | S_1 \cap \cdots \cap S_m$ does not vanish at the critical point a). Finally, A is given by

$$A = \frac{(2\pi)^{(n-m+1)/2} \rho}{\sqrt{D}}, \tag{2.3}$$

where ρ is the coefficient at the point a of the differential form

$$\omega \wedge dt_1 \wedge dt_2 \wedge \cdots \wedge dt_q = \rho dy_1 \wedge dy_2 \wedge \cdots \wedge dy_p,$$

and where D is the determinant of the matrix M' obtained by deleting one of the final m rows, and the corresponding column, of the following square matrix:

$$M = \begin{array}{r|c|c|c|}
 & \overbrace{1, 2, \ldots, p}^{j'=} & \overbrace{1, 2, \ldots, q}^{k'=} & \overbrace{1, 2, \ldots, m}^{i'=} \\ \hline
j = \left\{ 1, 2, \ldots, p \right. & \dfrac{\partial^2 l}{\partial y_j \partial y_{j'}} - \displaystyle\sum_{i=1}^{m} \lambda_i \dfrac{\partial^2 s_i}{\partial y_j \partial y_{j'}} & \dfrac{\partial t_{k'}}{\partial y_j} & \lambda_{i'} \dfrac{\partial s_{i'}}{\partial y_j} \\ \hline
k = \left\{ 1, 2, \ldots, q \right. & \dfrac{\partial t_k}{\partial y_{j'}} & \multicolumn{2}{c|}{} \\ \cline{1-2}
i = \left\{ 1, 2, \ldots, m \right. & \lambda_i \dfrac{\partial s_i}{\partial y_{j'}} & \multicolumn{2}{c|}{0} \\ \hline
\end{array}$$

The fact that this determinant D is independent of the row which has been deleted follows from relation (2.2). It also follows from the same relation that the upper left-hand block of the matrix M is a *tensor* with respect to the coordinates y. On the other hand, since the $\partial t_{k'}/\partial y_j$ and $\partial s_{t'}/\partial y_j$ are *vectors* with respect to the same coordinates, the variance of $\sqrt{D}$ is the same as the "unit volume" in the space Y, i.e., precisely the variance of the coefficient ρ. The coefficient A given by formula (2.3) is therefore a *scalar* in the coordinates y, and the same is true for the coordinates t, as is easily checked. One can also check that the expressions (2.1) are invariant under change of the local defining equation $l(t)$ for the Landau variety.

2.4 A practical calculation of the coefficient

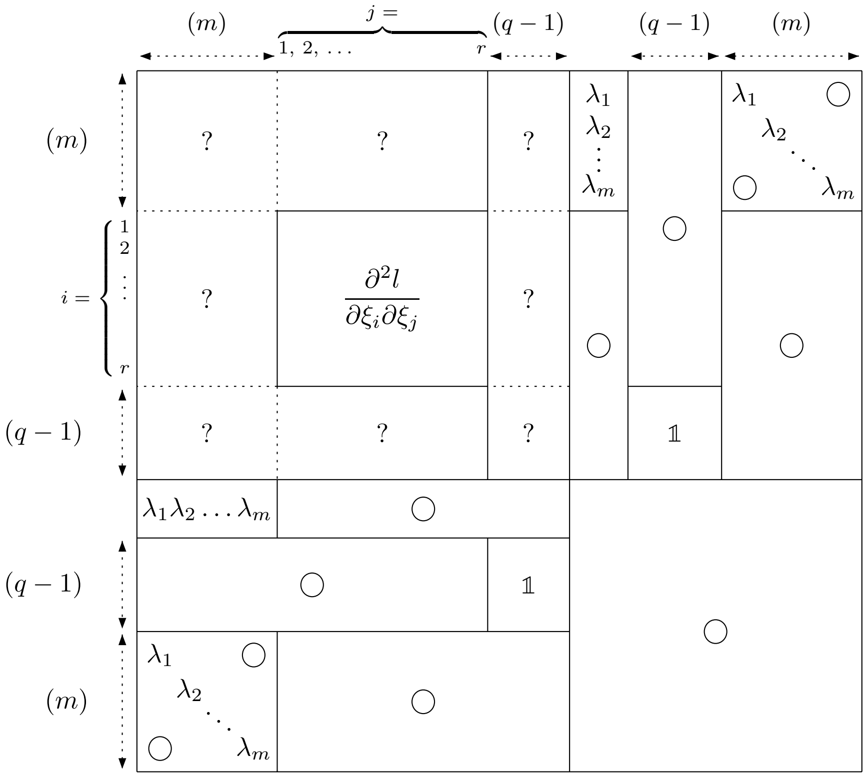
Let us choose coordinates $(y_1, y_2, \dots, y_p)$, $(t_1, t_2, \dots, t_q)$ such that

$$s_1(y) = y_1, \quad s_2(y) = y_2, \quad \dots, \quad s_m(y) = y_m, \quad l(t) = t_1,$$

and such that the projection $\pi : Y \to T$ can be written

$$t_1 = l(y), \quad t_2 = y_{n+2}, \quad \dots, \quad t_q = y_p \quad (p = n + q).$$

The matrix M can then be written:



where we have set

$$\xi_1 = y_{m+1}, \quad \xi_2 = y_{m+2}, \quad \dots, \quad \xi_r = y_{m+r}, \quad (r = n - m + 1).$$

Thus,

$$\sqrt{D} = \lambda_1 \lambda_2 \dots \lambda_m \sqrt{\operatorname{Det} \left\| \frac{\partial^2 l}{\partial \xi_i \partial \xi_j} \right\|}.$$

Note that $\xi_1, \xi_2, \dots, \xi_r$ can be interpreted as the coordinates of the manifold

$$(S_1 \cap S_2 \cap \dots \cap S_m)_\perp = Y_\perp \cap S_1 \cap S_2 \cap \dots \cap S_m,$$

where $Y_\perp$ is a submanifold of Y which is "*transverse*" to L [in the coordinates considered, $Y_\perp = Y|(t_2 = t_3 = \dots = t_q = 0)$]. $\operatorname{Det} \|\partial^2 l / \partial \xi_i \partial \xi_j\|$ is therefore the Hessian of the function l restricted to $(S_1 \cap S_2 \cap \dots \cap S_m)_\perp$ (in short, $\operatorname{Hess}_\perp l$), and the "simple pinching" condition is equivalent to the non-vanishing of D. Now let us calculate the coefficient ρ. By definition, it is the coefficient of the differential form

$$(\omega \wedge dl)_\perp = (\omega \wedge dl)|Y_\perp = \rho ds_1 \wedge \dots \wedge ds_m \wedge d\xi_1 \wedge \dots \wedge d\xi_r,$$

i.e.,

$$\left. \frac{(\omega \wedge dl)_\perp}{ds_1 \wedge \dots \wedge ds_m} \right|_{(S_1 \cap S_2 \cap \dots \cap S_m)_\perp} = \rho d\xi_1 \wedge \dots \wedge d\xi_r. \tag{2.5}$$

In summary,

$$A = \frac{(2\pi)^{(n-m+1)/2} \rho}{\lambda_1 \lambda_2 \dots \lambda_m \sqrt{\operatorname{Hess}_\perp l}}, \tag{2.6}$$

where ρ is given by (2.5).

The appeal of expression (2.6) is that it takes a very simple form in the two cases $m = n$ and $m = n + 1$:

The case $m = n + 1$: $r = 0$ and $\operatorname{Hess}_\perp l = 1$. *In the fibre Y_u*, one can calculate the *scalar*

$$\rho_{\widehat{n+1}} = \left. \frac{\omega}{ds_1 \wedge \dots \wedge ds_n} \right|_{(S_1 \cap \dots \cap S_n)_u}.$$

Hence, (2.5) can be written

$$\rho = \left. \frac{(\omega \wedge dl)_\perp}{ds_1 \wedge \dots \wedge ds_{n+1}} \right|_{(S_1 \cap \dots \cap S_n \cap S_{n+1})_\perp},$$

i.e., given (2.2),

$$\rho = \lambda_{n+1} \rho_{\widehat{n+1}}.$$

Therefore,

$$A = \frac{1}{\lambda_1 \lambda_2 \dots \lambda_n} \frac{\omega}{ds_1 \wedge ds_2 \wedge \dots \wedge ds_n}\bigg|_a. \tag{2.7}$$

The case $m = n$: $r = 1$, $\mathrm{Hess}_\perp l = \partial^2 l/\partial\xi^2$. We can choose the coordinate ξ in such a way that $l_\perp = \xi^2/2$, i.e., $\mathrm{Hess}_\perp l = 1$. In the fibre Y_t, for t close to u, $t \notin L$, the manifolds $S_{1t}, S_{2t}, \dots, S_{nt}$ are in general position, and we can therefore calculate the *scalar*

$$\sigma_t = \frac{\omega}{ds_1 \wedge \dots \wedge ds_n}\bigg|_{(S_1 \cap \dots \cap S_n)_i}.$$

Therefore, at the point t, we have the equalities

$$dl = \sum_{i=1}^n \lambda_i ds_i + dl_\perp = \sum_{i=1}^n \lambda_i ds_i + \xi d\xi,$$
$$(\omega \wedge dl)_\perp = \sigma_t ds_1 \wedge \dots \wedge ds_n \wedge \xi d\xi,$$

and, as a result,

$$\rho = \lim_{t \to u} \sigma_t \, \xi.$$

Thus,

$$A = \frac{\sqrt{2\pi}}{\lambda_1 \lambda_2 \cdots \lambda_n} \lim_{t \to u} \sqrt{2l(t)} \frac{\omega}{ds_1 \wedge \dots \wedge ds_n}\bigg|_{(S_1 \cap \dots \cap S_u)_t}. \tag{2.8}$$

2.9 The sign of the expressions (2.1)

The expressions (2.1) contain several signs which have not been defined: the sign of $\sqrt{l(t)}$ when $n + m - 1$ is odd; the sign of N, which depends on the orientation of the vanishing class; and the sign of A, which contains a square root $\sqrt{\mathrm{Hess}_\perp l_\alpha}$ [cf. (2.6)]. The interplay between the signs can be summarized in the following way: consider a value of t for which $l(t) > 0$. We therefore take $\sqrt{l(t)} > 0$, and the sign of NA is given by the following proposition (cf. [21], §22):

The manifold $(S_1 \cap S_2 \cap \dots \cap S_m)_\perp$ contains an oriented ball, whose oriented boundary is the vanishing sphere $e(S_1 \cap S_2 \cap \dots \cap S_m)$, and on which $\sqrt{\mathrm{Hess}_\perp l}\, d\xi_1 \wedge d\xi_2 \wedge \dots \wedge d\xi_r$ defines a *positive measure*.

2.10 Proof of the formulae (2.1′)–(2.1$^{\mathrm{v}}$) (J. Leray)

It will be convenient to assume, as in the proof of proposition 1.8, that

$$T = T^{(1)} \times T^{(2)} \times \dots \times T^{(m)},$$

where each s_i *only depends on* $t^{(i)} \in T^{(i)}$, and does indeed depend upon it. Furthermore, *in a neighbourhood of the point* u, we will equip each $T^{(i)}$ with an *affine* structure such that $s_i(x, t^{(i)})$ is a *linear* function of $t^{(i)}$, for $x \in U$. We will call a "*homogeneous polynomial of degree* β", with $\beta = \{\beta_1, \beta_2, \ldots, \beta_m\}$ (β_i integers $\geqslant 0$),[2] a product $P_\beta(t) = \prod_{i=1}^m P_{\beta_i}(t^{(i)})$ of homogeneous polynomials $P_{\beta_i}(t^{(i)})$ of degree β_i. Such polynomials will occur either as polynomials of derivations $P_\beta\,(\partial/\partial t)$, or in expressions of the form $P_\beta(\dot{s})$, which is shorthand for $\prod_{i=1}^m P_{\beta_i}\left(\partial s_i/\partial t^{(i)}\right)$. By the linearity condition, $P_\beta(\dot{s})$ is a function of x only (for $x \in U$), which we will assume to be non-zero on the open set U.

First step: construction of auxiliary functions. Let $R_{\beta,\gamma} = Q_\beta/Q_\gamma$ be a rational function, which is a quotient of homogeneous polynomials of degree β and γ. Assuming for the sake of argument that

$$\begin{aligned} &\beta_j - \gamma_j \leqslant 0 \quad (j = 1, 2, \ldots, v),\\ &\beta_j - \gamma_j > 0 \quad (j = v+1, \ldots, m), \end{aligned}$$

we set

$$(2.11) \quad j_{R_{\beta,\gamma}}(t) = \int_{e^{\nu+1\ldots m}} \frac{d^{\sum_{k=\nu+1}^m (\beta_k - \gamma_{k-1})}}{ds_{\nu+1}^{\beta_{\nu+1}-\gamma_{\nu+1}} \wedge \cdots \wedge ds_m^{\beta_m - \gamma_m}} \times \left[\left(\prod_{j=1}^{\nu} \frac{(-s_j)^{\gamma_j - \beta_j}}{(\gamma_j - \beta_j)!}\right) R_{\beta,\gamma}(-\dot{s})\omega'\right],$$

where ω' is a holomorphic function of degree n which is independent of t, just like ω in formula (2.0), and will be specified later. For every homogeneous polynomial P, we have the *derivation formula*[3]

$$(2.12) \qquad \boxed{P\left(\frac{\partial}{\partial t}\right) j_R(t) = j_{PR}(t).}$$

Let us evaluate $j_1(t)$. Obviously,

$$j_1(t) = \int_{\boldsymbol{e}} \omega' \simeq \rho' \operatorname{Meas} \boldsymbol{e},$$

where ρ' is the coefficient of the form $\omega' = \rho' dx_1 \wedge \cdots \wedge dx_n$, evaluated at the pinching point a. Meas $\boldsymbol{e}$ is the measure of the vanishing cell $\boldsymbol{e}$, which easily evaluated in the local coordinates of chapter V (§2.1). In fact, it is the area swept out by the vanishing cell $\boldsymbol{e}^m$ as t_1 increases, starting from zero, i.e.:

$$\begin{aligned} \operatorname{Meas} \boldsymbol{e} = \operatorname{Primitive} \operatorname{Meas} \boldsymbol{e}^m &= (\mathrm{Prim})^2 \operatorname{Meas} \boldsymbol{e}^{m-1,m}\\ &= \cdots = (\mathrm{Prim})^{m-1} \operatorname{Meas} \boldsymbol{e}^{2\ldots m}. \end{aligned}$$

[2] The letter β will also denote the integer $\sum_{i=1}^m \beta_i$.

[3] Which we admit here without proof: it can be proved immediately by iterating the derivation formulae (10.5) and (10.6) and Leray's formula [20].

Now, Meas $\boldsymbol{e}^{2\dots m}$ is the volume of a ball of radius $\sqrt{t_1}$ in $(n-m+1)$-dimensional space, which is equal to

$$\text{Meas}\, \boldsymbol{e}^{2\dots m} = \frac{\pi^{(n-m+1)/2}|t_1|^{(n-m+1)/2}}{\Gamma\left(1+\frac{n-m+1}{2}\right)},$$

so that

$$\text{Meas}\, \boldsymbol{e} = \frac{\pi^{(n-m+1)/2}|t_1|^{(n+m-1)/2}}{\Gamma\left(1+\frac{n+m-1}{2}\right)}.$$

As a result, in these coordinates we have

$$j_1(t) \simeq \frac{\pi^{(n-m+1)/2}\rho'|t_1|^{(n+m-1)/2}}{\Gamma\left(1+\frac{n+m-1}{2}\right)},$$

which can be rewritten in general coordinates as

$$j_1(t) = \frac{A'[l(t)]^{(n+m-1)/2}}{\Gamma\left(1+\frac{n+m-1}{2}\right)}(1+0(t)),\text{[4]} \tag{2.13}$$

where A' is given by expression (2.3), with ω replaced by ω'.

2.14 Lemma. *$j_{R_{\beta,\gamma}}(t)[l(t)]^{\beta-\gamma-\frac{n+m-1}{2}}$ is holomorphic at the point u.*

It suffices to prove this for $\beta = 0$, since the general case follows from this by the derivation formula (2.12). Now,

$$j_{R_{0,\gamma}}(t) = \int_{\boldsymbol{e}} \left(\prod_{i=1}^{m} \frac{(-s_i)^{\gamma_i}}{\gamma_i!}\right) R_{0,\gamma}(-\dot{s})\omega'$$

and we have seen that Meas $\boldsymbol{e} <$ Cst $|l(t)|^{(n+m-1)/2}$. Since, clearly, $|s_i| <$ Cst $|l(t)|$, we deduce that

$$|j_{R_{0,\gamma}}(t)| < \text{Cst}\, |l(t)|^{\gamma+\frac{n+m-1}{2}},$$

and the function $j_{R_{0,\gamma}}(t) = [l(t)]^{-\gamma-\frac{n+m-1}{2}}$ is of bounded modulus. On the other hand, it is single-valued, because

$$\omega_* \boldsymbol{e} = \begin{cases} \boldsymbol{e}, & \text{if } n+m-1 \text{ is odd,} \\ -\boldsymbol{e}, & \text{if } n+m-1 \text{ is even} \end{cases}$$

(cf. chap. V, §2.6), and is therefore holomorphic (by Riemann's theorem).

2.15 Lemma. *If $n+m-1$ is even, $j_{R_{\beta,\gamma}}(t)$ is holomorphic.*

[4] Observe that this formula is also valid in the case $m = n+1$.

In this case, all the vanishing classes are *invariant* in the sense of chapter V, §2.7, which means that we can apply lemma 1.2.

$$
(2.16)\quad \begin{cases} j_{R_{\beta,\gamma}}(t) = \dfrac{N\left(\prod_{i=1}^{m} \lambda_i^{\beta_i-\gamma_i}\right) R_{\beta,\gamma}(\dot{s})}{\Gamma\left(1+\frac{n+m-1}{2}-\beta+\gamma\right)}\,[l(t)]^{\frac{n+m-1}{2}-\beta+\gamma}(1+0(t)),\\ \textit{except in the case } n+m-1 \textit{ even} < 2(\beta-\gamma). \end{cases}
$$

By the derivation formula (2.12), we have $j_1(t) = Q_\gamma(\frac{\partial}{\partial t}) j_{1/Q_\gamma}(t)$. Integrating formula (2.13), and taking into account lemma 2.14, gives:

$$
j_{1/Q_\gamma}(t) = \frac{N\left(\prod_{i=1}^{m} \lambda_i - \gamma_i\right)[l(t)]^{\frac{n+m-1}{2}+\gamma}}{Q_\gamma(\dot{s})\Gamma\left(1+\frac{n+m-1}{2}+\gamma\right)}(1+0(t)).
$$

By applying the polynomial operator $Q_\beta(\partial/\partial t)$ to this expression, we obtain (2.16), except in the case $n+m-1$ even $< 2(\beta-\gamma)$, where the power of the leading term becomes zero after a certain number of differentiations.

Second step: comparison between the integral to be studied with auxiliary functions. We set $R = Q^+/Q^-$, where:

Q^+ is a homogeneous polynomial of degree $(0,0,\dots,0,\alpha_{\mu+1},\dots,\alpha_m)$,

Q^- is a homogeneous polynomial of degree $(-\alpha_1,\dots,-\alpha_\mu,0,\dots,0)$,

and let us choose the form ω' which is used to define the auxiliary functions of the first step to be

$$
\omega' \frac{\omega}{R(-\dot{s})}.
$$

Thus,

$$
\begin{aligned}
j_R(t) &= \int_{e^{\mu+1\dots m}} \frac{d^{\sum_{k=\mu+1}^{m}(\alpha_k-1)}}{ds_{\mu+1}^{\alpha_{\mu+1}}\wedge\dots\wedge ds_m^{\alpha_m}}\left[\left(\prod_{j=1}^{\mu}\frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!}\right)\omega\right]\\
&= \frac{(\alpha_{\mu+1}-1)!\cdots(\alpha_m-1)!}{(2\pi i)^{m-\mu}}\int_{e_{1.2\dots\mu}}\left(\prod_{j=1}^{\mu}\frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!}\right)\frac{\omega}{s_{\mu+1}^{\alpha_{\mu+1}}\cdots s_m^{\alpha_m}}
\end{aligned}
$$

by the residue theorem.

2.17 Theorem. *If $n-m+1$ is odd, the function*

$$
J_\alpha(t) + \frac{N}{2}\frac{(2\pi i)^{m-\mu}}{(\alpha_{m+1}-1)!\cdots(\alpha_m-1)!} j_R(t)
$$

is holomorphic at the point u.

This is because the class $2\widetilde{h}_{1.2\dots\mu} + N\widetilde{e}_{1.2\dots\mu}$ is invariant (cf. chap. V, lemma 2.10).

Together with (2.16), this theorem proves formula (2.1′).

2.18

Now let us concentrate on the case where $n+m-1$ *is even*. The integration cycle $\widetilde{\varGamma}_{1.2\ldots\mu}$ can be assumed to cut the sphere $\dot{U}$, which is the boundary of the ball U, *transversally*: this gives a decomposition

$$\widetilde{\varGamma}_{1.2\ldots\mu} = \underset{\substack{\cap\\ U}}{{}'\widetilde{\varGamma}_{1.2\ldots\mu}} + \underset{\substack{\cap\\ X-U}}{{}''\widetilde{\varGamma}_{1.2\ldots\mu}}.$$

In this case, one can show that ${}'\widetilde{\varGamma}_{1.2\ldots\mu}$ is homologous to $N\delta_{\mu+1}\circ\cdots\circ\delta_m\gamma^{\mu+1\ldots m}$ in $U-S_{\mu+1\ldots m}$, modulo $\dot{U}\cup S_{1.2\ldots\mu}$, where $\gamma^{\mu+1\ldots m}$ is a chain of $U\cap S^{\mu+1\ldots m}$ bounded by $S_{1.2\ldots\mu}$. As $l(t)$ winds around zero, this chain $\gamma^{\mu+1\ldots m}$ varies continuously, and its measure remains *bounded* as long as $l(t)$ only winds a *finite* number of times around zero.

2.19 Lemma. *Let P be a homogeneous polynomial of degree* $(0,\ldots,0,\alpha_{\mu+1}-1,$ $\ldots,\alpha_m-1)$.

The function

$$J_\alpha(t) - \frac{1}{(\alpha_{\mu+1}-1)!\cdots(\alpha_m-1)!P(-\dot{s})}P\left(\frac{\partial}{\partial t}\right) \times\int_{{}'\widetilde{\varGamma}_{1.2\ldots\mu}}\left(\prod_{j=1}^{\mu}\frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!}\right)\frac{\omega}{s_{\mu+1}\cdots s_m}$$

is holomorphic at the point u.

To see this, note that the polynomial of derivations $P\left(\partial/\partial t\right)$ only involves the $\partial/\partial t^{(k)}$ (where $k=\mu+1,\ldots,m$) so we can keep the variables $t^{(j)}$ ($j=1,2,\ldots,\mu$), and hence the corresponding manifolds S_j, *fixed*. The analysis of section 1 therefore shows that the integral over ${}''\varGamma_{1.2\ldots\mu}$ is holomorphic, and so the function

$$J_\alpha(t) - \int_{{}'\widetilde{\varGamma}_{1.2\ldots\mu}}\left(\prod_{j=1}^{\mu}\frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!}\right)\frac{\omega}{s_{\mu+1}^{\alpha_{\mu+1}}\cdots s_m^{\alpha_m}}$$

is holomorphic. To see that this function is indeed the same one as in lemma 2.19, it is enough to observe, as in the proof of lemma 1.2, that the chain ${}'\widetilde{\varGamma}_{1.2\ldots\mu}$ can be taken *independently of t* for small variations in t, which justifies the differentiation under the integral.

(2.20)[5] *We have the identity*

$$\int_{'\widetilde{\Gamma}_{1.2\ldots\mu}} \left(\prod_{j=1}^{\mu} \frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!} \right) \frac{\omega}{s_{\mu+1} \cdots s_m}$$
$$= \int_{^\circ\Gamma} \text{the same integrand}$$
$$+ (2\pi i)^{m-\mu} N \int_{\boldsymbol{\gamma}^{\mu+1\ldots m}} \frac{1}{ds_{\mu+1} \wedge \cdots \wedge ds_m} \left[\left(\prod_{j=1}^{\mu} \frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!} \right) \omega \right],$$

where $^\circ\Gamma$ *is a chain of* $\dot{U}$ *which intersects* $S_{\mu+1}, \ldots, S_m$ *in general position* (Fig. VI.2). Thus, §2.18 can be reformulated as follows:

$$'\widetilde{\Gamma}_{1.2\ldots\mu} = N\,\delta^{\varepsilon}_{\mu+1} \circ \cdots \circ \delta^{\varepsilon}_{m} \boldsymbol{\gamma}^{\mu+1\ldots m} + {}^{\varepsilon}\Gamma + \text{a chain of } S_{1.2\ldots\mu} + \text{a boundary},$$

where ${}^{\varepsilon}\Gamma$ is a chain of $\dot{U} - S_{\mu+1\ldots\mu}$ and the index ε above δ denotes the radius of the tubular neighbourhood which is used to define the coboundary δ. Only the first two terms of this sum give an integral which is non-zero: *as* $\varepsilon \to 0$, the integral over the first term can be calculated by taking residues (it would not be correct to apply the residue theorem for finite ε, because $\boldsymbol{\gamma}^{\mu+1\ldots m}$ *is not a cycle*). From this, we deduce formula (2.16), with $^\circ\Gamma = \lim_{\varepsilon\to 0} {}^{\varepsilon}\Gamma$. To see that the integral over $^\circ\Gamma$ converges, it suffices to observe that every $S_k \cap {}^\circ\Gamma$ is of real codimension 2 in $^\circ\Gamma$, whereas the s_k appear with a power 1 in the denominator.

2.21 Lemma. *If* $n + m - 1$ *is even, the function*

$$f_\alpha(t) = \int_{'\widetilde{\Gamma}_{1.2\ldots\mu}} \Big(\prod_{j=1}^{\mu} \frac{(-s_j)^{-\alpha_j}}{(-\alpha_j)!} \Big) \frac{\omega}{s_{\mu+1} \cdots s_m}$$
$$- (2\pi i)^{m-\mu-1} N P(-\dot{s}) j_{R/P}(t) \log l(t)$$

is holomorphic at the point u *(where* $\deg P = (0, \ldots, 0, \alpha_{\mu+1} - 1, \ldots, \alpha_m - 1)$*).*

By the Picard–Lefschetz formula,

$$\operatorname{Var} {}'\widetilde{\Gamma}_{1.2\ldots\mu} = N\,\delta_{\mu+1} \circ \cdots \circ \delta_m \boldsymbol{e}^{\mu+1\ldots m},$$

which proves, using the residue theorem, the definition of $j_{R/P}(t)$, and lemma 2.15, that the function $f_x(t)$ is single-valued. Let us bound it above: its second term is bounded above by $\mathrm{Cst}\,|\log l(t)|$ by virtue of lemma 2.15, whereas its first term is given by formula (2.20). Now, in formula (2.20), the integral over $^\circ\Gamma$ converges absolutely and uniformly, and is therefore bounded.

[5] From §(2.20) to 2.23, we exclude the case ($\mu = 0$, $m = n + 1$).

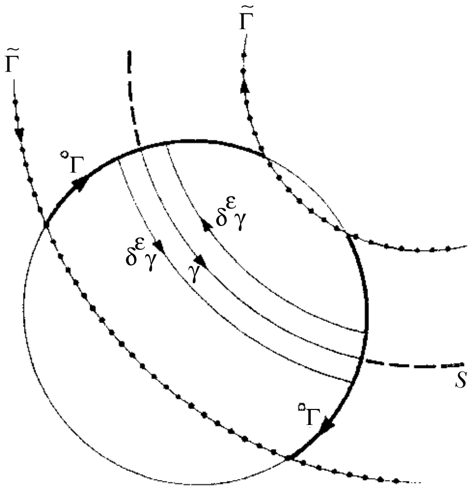


Fig. VI.2.

As for the integral over $\gamma^{\mu+1\dots m}$, its integrand is bounded above by a constant if $\mu > 0$, and by $\mathrm{Cst}/\sqrt{|l(t)|}$ if $\mu = 0$ (to see this, use local coordinates).

In summary,

$$|f_\alpha(t)| < \mathrm{Cst}\,|\log l(t)| + \frac{\mathrm{Cst}}{\sqrt{|l(t)|}},$$

which is enough to prove that $f_\alpha(t)$ is holomorphic, since it is single-valued.

2.22 Theorem. *Let P be a homogeneous polynomial of degree β. If $n+m-1$ is even, the function*

$$F_\alpha(t) = J_\alpha(t) - \frac{(2\pi i)^{m-\mu-1}N}{(\alpha_{\mu+1}-1)!\cdots(\alpha_m-1)!}P\left(\frac{\partial}{\partial t}\right)[j_{R/P}(t)\log l(t)]$$

is holomorphic, provided that $\beta \geqslant \alpha - \frac{n+m-1}{2}$.

In the special case $\beta = (0,\dots,0,\alpha_{\mu+1}-1,\dots,\alpha_m-1)$, this theorem is an immediate consequence of lemmas 2.19 and 2.21. The general case follows from it, by the following lemma.

2.23 Lemma. *If P and P' are two homogeneous polynomials such that $P(-\dot{s})$ $P'(-\dot{s}) \neq 0$, and if*

$$\textit{degree } P = \beta \geqslant \alpha - \frac{n+m-1}{2},$$

then the function

$$P\left(\frac{\partial}{\partial t}\right)P'\left(\frac{\partial}{\partial t}\right)[j_{R/PP'}(t)\log l(t)] - P\left(\frac{\partial}{\partial t}\right)[j_{R/P}(t)\log l(t)]$$

is holomorphic at the point u.

To prove lemma 2.23, it suffices to recall that $j_{R/PP'}(t)$ has a zero of order at least $\beta + \beta' - \alpha\frac{n+m-1}{2}$ on L (cf. lemma 2.14), i.e., a zero of order $\geqslant \beta' =$ degree P', so that

$$P'\left(\frac{\partial}{\partial t}\right)[j_{R/PP'}(t)\log l(t)] - j_{R/P}(t)\log l(t)$$

is holomorphic.

We have thus proved theorem 2.22*. Together with* (2.6)*, this proves formula* (2.1$''$) *and* (2.1$'''$).

It remains to prove two special cases:

2.24 Proof of (2.1$^{\mathrm{iv}}$)

$$\mu = m = n+1, \quad J_\alpha(t) = \int_{\boldsymbol{h}} \prod_{i=1}^{n+1} \frac{[-s_i(x,t)]^{-\alpha_i}}{(-\alpha_i)!}\omega.$$

By the Picard–Lefschetz formula, $\boldsymbol{h}$ is unramified, and the function $J_\alpha(t)$ is therefore *uniform*. But it is obviously *bounded*, and therefore *holomorphic*.

2.25 Proof of (2.1$^{\mathrm{v}}$)

$$\mu = 0, \quad m = n+1, \quad J_\alpha(t) = \int_{\widetilde{h}} \frac{\omega}{s_1^{\alpha_1}\cdots s_n^{\alpha_n} s_{n+1}^{\alpha_{n+1}}}.$$

By chapter V, §2.12, the class

$$\widetilde{h} + (-)^n N\,\delta_1 \circ \delta_2 \circ \cdots \circ \delta_n \widehat{\mathscr{O}^{n+1}}$$

is invariant. Therefore

$$J_\alpha(t) = \text{hol. fn.} + (-)^{n+1} N \int_{\delta_1\circ\delta_2\circ\cdots\circ\delta_n \widehat{\mathscr{O}^{n+1}}} \frac{\omega}{s_1^{\alpha_1}\cdots s_n^{\alpha_n} s_{n+1}^{\alpha_{n+1}}}.$$

By transforming the second integral as in lemma 2.19, and then by applying the residue theorem, we find that

$$\begin{aligned}
J_\alpha(t) &= \text{hol. fn.} + \frac{(-)^{n+1}N}{(\alpha_1-1)!\cdots(\alpha_n-1)!(\alpha_{n+1}-1)!P(-\dot{s})} \\
&\qquad \times P\left(\frac{\partial}{\partial t}\right)\int_{\delta_1\circ\cdots\circ\delta_n \widehat{\mathscr{O}^{n+1}}} \frac{\omega}{s_1\cdots s_n s_{n+1}} \\
&= \text{hol. fn.} + \frac{(-)^{n+1}N(2\pi i)^n}{(\alpha_1-1)!\cdots(\alpha_n-1)!(\alpha_{n+1}-1)!P(-\dot{s})} \\
&\qquad \times P\left(\frac{\partial}{\partial t}\right) \left.\frac{\frac{\omega}{s_{n+1}}}{ds_1\wedge\cdots\wedge ds_n}\right|_{\widehat{\mathscr{O}^{n+1}}}
\end{aligned}$$

where P is a homogeneous polynomial of degree $(\alpha_1 - 1, \dots, \alpha_n - 1, \alpha_{n+1} - 1)$.

Now, in the local coordinates of chapter V, §2.1,

$$s_{n+1}\big(\widehat{\mathscr{O}^{n+1}}\big) = t_1,$$

i.e., in general coordinates

$$s_{n+1}\big(\widehat{\mathscr{O}^{n+1}}\big) = \frac{l(t)}{\lambda_{n+1}}$$

and therefore

$$J_\alpha(t) = \text{hol. fn.}$$
$$+ \frac{(-)^{n+1} N (2\pi i)^n \left(\prod_{i=1}^{n+1} \lambda_i^{\alpha_1 - 1}\right)}{(\alpha_1 - 1)! \cdots (\alpha_{n+1} - 1)!} \lambda_{n+1} \frac{\omega}{ds_1 \wedge \cdots \wedge ds_n}\bigg|_a \frac{(\alpha - n - 1)!}{[l(t)]^{\alpha - n}}.$$

VII

Ramification of an integral whose integrand is itself ramified

The ramification problem for an integral has already been solved when the integrand has polar singularities along $S_1, S_2, \ldots, S_m$: by applying the Picard–Lefschetz formula and the residue theorem, we showed that a "simple loop" ω_L around the Landau variety L changes the integral $J(t) = \int_{\widetilde{\Gamma}} \varphi_t$ by

$$\omega_L^* J(t) = J(t) + N \int_{\widetilde{e}} \varphi_t = J(t) + N(2\pi i)^m \int_e \operatorname{Res}^m \varphi_t.$$

In this chapter we will consider an analogous problem where the integrand φ_t is itself "ramified" around $S_1, S_2, \ldots, S_m$: in this case, we must no longer work in the space $Y - S$, but in a "*covering space*" $\widetilde{Y - S}$ of $Y - S$. Let us begin with some general comments on covering spaces.

1 Generalities on covering spaces

A *covering space* of a topological space[1] Y is a locally trivial fibre bundle $\widetilde{Y} \xrightarrow{r} Y$ with *discrete* fibres F.

1.1

Let $F_y = r^{-1}(y)$ denote the fibre above y. We show, exactly as in §IV.2.6, that *every path λ in Y with initial point a, and end point b, defines an isomorphism*

$$\lambda_\bullet : F_a \longrightarrow F_b$$

which only depends on the homotopy class of the path. Let us make this isomorphism more explicit: if $\widetilde{a} \in F_a$, there exists a unique path $\widetilde{\lambda}_{\widetilde{a}}$ in $\widetilde{Y}$ with initial point $\widetilde{a}$, such that $r \circ \widetilde{\lambda}_{\widetilde{a}} = \lambda$. The end point $\widetilde{b}$ of this path is the point $\lambda \cdot \widetilde{a}$.

[1] In the sequel, we must assume that Y is locally compact and paracompact.

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_7,

1.2

Setting $a = b$ in §1.1 defines a homomorphism

$$\chi : \pi_1(Y, a) \longrightarrow \operatorname{Aut} F_a \quad \text{via } \chi(\lambda) = \lambda_\bullet.$$

The *quotient group* $\Omega_a = \pi_1(Y, a)/\operatorname{Ker}\chi$ (which is isomorphic to the image of χ in $\operatorname{Aut} F_\alpha$) is called the *structure group of the covering space*. This group is independent of the base point a, up to isomorphism.

1.3

The projection $r : \widetilde{Y} \to Y$ induces a homomorphism of fundamental groups

$$r_{\widetilde{a}} : \pi_1\left(\widetilde{Y}, \widetilde{a}\right) \longrightarrow \pi_1(Y, a) \quad [\text{where } a = r(\widetilde{a})],$$

defined by $r_{\widetilde{a}}(\widetilde{\lambda}) = r \circ \widetilde{\lambda}$.

This homomorphism is *injective*, because if the projection λ of a loop $\widetilde{\lambda}$ in $\widetilde{Y}$ is homotopic to zero, it is homotopic to zero itself.[2]

1.4

Let us set $\pi_1^{\widetilde{a}} = \operatorname{Im} r_{\widetilde{a}}$. This is the subgroup of $\pi_1(Y, a)$ consisting of the loops λ whose "lift" $\widetilde{\lambda}_{\widetilde{a}}$ (using the notation of §1.1) ends at the point $\widetilde{a}$, which clearly shows that

$$\bigcap_{\widetilde{a} \in F_a} \pi_1^{\widetilde{a}} = \operatorname{Ker}\chi.$$

In everything that follows, the space $\widetilde{Y}$ will be assumed to be path connected (and locally path connected).

1.5

The set $\pi_1(Y, a)/\pi_1^{\widetilde{a}}$ of *left cosets of $\pi_1^{\widetilde{a}}$ in $\pi_1(Y, a)$ is isomorphic to the fibre F_a.* To show this, let $\widetilde{a} : \pi_1(Y, a) \to F_a$ be the map which sends the loop λ to the point $\widetilde{b} = \lambda \cdot \widetilde{a}$. This map is obviously surjective, because for all $\widetilde{b} \in F_a$, there exists a loop $\widetilde{\lambda}$ in $\widetilde{Y}$ which joins $\widetilde{a}$ to $\widetilde{b}$, and it is enough to set $\lambda = r \circ \widetilde{\lambda}$. Now let λ and λ' be two loops such that $\lambda \cdot \widetilde{a} = \lambda' \cdot \widetilde{a} = \widetilde{b}$. This means that the two paths $\widetilde{\lambda}_{\widetilde{a}}$ and $\widetilde{\lambda}'_{\widetilde{a}}$ have the same end point $\widetilde{b}$, and therefore $\widetilde{\lambda}_{\widetilde{a}}^{-1} \cdot \widetilde{\lambda}'_{\widetilde{a}}$ is a loop based at $\widetilde{a}$, which projects onto $\lambda^{-1} \cdot \lambda'$, and therefore $\lambda^{-1} \cdot \lambda' \in \pi_1^{\widetilde{a}}$.

[2] This is a special case of the general "*homotopy lifting*" property for fibre bundles: it is this property which is used to prove that every bundle over a contractible base is trivial.

1.6

If $\widetilde{a}$ and $\widetilde{b} \in F_a$, the subgroups $\pi_1^{\widetilde{a}}$ and $\pi_1^{\widetilde{b}}$ are mapped to each other by an inner automorphism of $\pi_1(Y, a)$. To see this, let $\widetilde{\lambda}$ be a path with initial point $\widetilde{a}$ and end point $\widetilde{b}$, and let $\lambda = r \circ \widetilde{\lambda}$. The commutativity of the diagram

$$\begin{tikzcd}
\pi_1\left(\widetilde{Y}, \widetilde{a}\right) \arrow[r, "r_{\widetilde{a}}"] \arrow[d, "{[\widetilde{\lambda}]}", "\wr"'] & \pi_1(Y, a) \arrow[d, "{[\lambda]}", "\wr"'] \\
\pi_1\left(\widetilde{Y}, \widetilde{b}\right) \arrow[r, "r_{\widetilde{b}}"] & \pi_1(Y, a)
\end{tikzcd}$$

(in which the vertical arrows are isomorphisms) proves that $\operatorname{Im} r_{\widetilde{a}}$ and $\operatorname{Im} r_{\widetilde{b}}$ are mapped to each other by the automorphism $[\lambda]$.

1.7 Regular covering spaces

A covering space is called "*regular*" if $\pi_1^{\widetilde{a}}$ is a *normal* subgroup of $\pi_1(Y, a)$. This subgroup is therefore independent of $\widetilde{a} \in F_a$ (by §1.6), and it coincides with $\operatorname{Ker} \chi$ (by §1.4). As a result, *the structural group Ω_a is isomorphic to the fibre F_a*, by §1.5, and acts transitively upon it by *left multiplication.* Having chosen the point $\widetilde{a}$ which occurs in §1.5 once and for all, we can *identify* the fibre F_a with the group Ω_a, which also allows us to define *right* multiplication of a point of F_a by an element of Ω_a. We will see that this right multiplication canonically extends to all the fibres F_b, so that Ω_a *acts on the whole covering space $\widetilde{Y}$ as a group of operators acting on the right which respect the projection r*. To make this explicit, let $\omega \in \Omega_a$. The right operator $\cdot\,\omega$ maps the point $\widetilde{b} \in \widetilde{Y}$ to the point $\widetilde{b} \cdot \omega = \lambda \cdot ((\lambda^{-1} \cdot \widetilde{b}) \cdot \omega)$, where λ is any path which joins a to $b = r(\widetilde{b})$, and $\lambda^{-1} \cdot \widetilde{b}$ is a point of F_a which is mapped by right multiplication by ω to a point $(\lambda^{-1} \cdot \widetilde{b}) \cdot \omega$, which is finally sent into F_b by the operator $\lambda\,\cdot$. One immediately checks that the point obtained does not depend on the choice of path λ.

1.8 Construction of all possible covering spaces of a space Y

One proves that, to every subgroup $\pi_1^{\bullet}$ of $\pi_1(Y, a)$ there corresponds a covering $\widetilde{Y} \xrightarrow{r} Y$, which is unique up to an isomorphism of fibre bundles, such that $r_{\widetilde{a}}(\pi_1(\widetilde{Y}, \widetilde{a})) = \pi_1^{\bullet}$. The space $\widetilde{Y}$ can be constructed as the set of homotopy classes of paths in Y, by identifying paths which are equivalent modulo $\pi_1^{\bullet}$.

In particular, if $\pi_1^{\bullet} = 0$, the covering space obtained in this way is called the *universal covering space* of Y: it is the only simply connected covering space of Y, and it "covers" all the others.

2 Generalized Picard–Lefschetz formulae

2.1

We will use the notations and hypotheses of chapter V, but we will focus on the homology of $\widetilde{X-S_t} \approx (\widetilde{Y-S})_t$ rather than $X - S_t \approx (Y-S)_t$, where $\widetilde{Y-S}$ is a *covering space* of $Y-S$, $(\widetilde{Y-S})_t = \widetilde{Y-S} \cap \widetilde{\pi}^{-1}(t)$, and the projection map $\widetilde{\pi}$ is defined by the commutative diagram:

$$\begin{tikzcd} \widetilde{Y-S} \arrow[r, "r"] \arrow[dr, "\widetilde{\pi}"'] & Y-S \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

Observe that if $Y-S$ is a locally trivial fibre bundle with base T, the same will be true for $\widetilde{Y-S}$, by the very definition of a "covering space". As a result, the Landau varieties of our new problem will be the same as those in chapter V. In the same way as in chapter V, the ramification problem around these Landau varieties can be *localized* to an open set $\widetilde{V-S} = \widetilde{Y-S}|V$, where V is a small ball neighbourhood of a pinching point $a \in S_1 \cap S_2 \cap \cdots \cap S_m$.

The structure of the covering space $\widetilde{V-S}$ could clearly not be much simpler: in Y, the submanifolds $S_1, S_2, \ldots, S_m$ intersect each other in general position, so that $V-S$ has the homotopy type of a product of m circles:

$$V - S \approx (\mathbb{C} - \{0\})^m \times \mathbb{C}^{p-m}.$$

In this case, we know that the fundamental group $\pi_1(V-S)$ is the *free commutative group* generated by the "small loops" $\omega_1, \omega_2, \ldots, \omega_m$ which wind around $S_1, S_2, \ldots, S_m$ respectively. Every covering space of $V-S$ is therefore *regular*, and as a result, $\pi_1(V-S)$ acts as a *group of automorphisms* on the covering space (we will view it as a group of *left* automorphisms: since it is commutative, its right action, which was defined in §1.7, coincides with its left action, which was defined in §1.2).[3]

2.2 Description of the vanishing chains

As in chapter V, let us choose a point $t \notin L$, and set $U = V_t$.

The *vanishing cell* $\boldsymbol{e}$ of §V.2.2 can be viewed as defining a (non-compact) "cell" in $U-S$, whose inverse image $r^{-1}(\boldsymbol{e})$ in $\widetilde{U-S} = \widetilde{Y-S}|(U-S)$ is

[3] We should point out that in reference [29], special importance is given to those covering spaces $\widetilde{V-S}$ for which $\omega_1, \omega_2, \ldots, \omega_m$ act "independently", i.e., those for which the subgroup $\pi_1^\bullet \subset \pi_1(V-S)$ can be spanned by the loops of the form $\omega_1^{\nu^1}, \omega_2^{\nu^2}, \ldots$, without "cross terms" such as $\omega_1\omega_2, \ldots$

a union of "cells" ${}_f(\boldsymbol{e})$, $f \in F$ (where the discrete set F is the fibre of the covering space). To each of these "cells" ${}_f(\boldsymbol{e})$ we will associate the "cycle"[4]

$$ {}_f e = (1-\omega_1)(1-\omega_2)\cdots(1-\omega_m)_* \, {}_f \boldsymbol{e}. $$

We can construct a deformation of this "cycle" which *moves it away* from S [29], and transforms it into a *cycle with compact support* in $\widetilde{U-S}$, denoted ${}_f\widetilde{e}$, and whose projection $\widetilde{e} = r_* {}_f\widetilde{e}$ belongs to the vanishing class $\widetilde{e}(U-S)$ of chapter V. Fig. VII.1 illustrates, in the case $n = m = 1$, the transition from the "cycle" ${}_f e = (1-\omega)_* {}_f\boldsymbol{e}$ to the cycle ${}_f\widetilde{e}$. Here, ω denotes the unique generator of the structural group and can be represented in $U-S$ by a small loop around *any one* of the two points which make up S_t. We have used two different styles (full and broken lines) for the cells situated in two different "sheets" (the cut which separates these two leaves has been chosen arbitrarily, and is indicated by the hatched region in the diagram).

Fig. VII.1.

In the general case (where n, m are arbitrary), one can get an intuitive idea of the situation by observing that the loop ω_i "does not act" on the "i-th boundary" of the "cell" ${}_f\boldsymbol{e}$ (by the "i-th boundary" we mean the part of its "boundary" situated in S_i).[5] The "cycle" ${}_f e$ defined above "therefore has a non-zero boundary in S", and so it is not surprising that we can move it away from S to make it into a *cycle* of $\widetilde{U-S}$.

2.3 Picard–Lefschetz formulae

The variation of a homology class $\widetilde{h} \in H_n(X - S_t)$ is given by the formula[6]

$$ (\widetilde{P}) \qquad \boxed{\operatorname{Var}\widetilde{h} = \sum_{f\in F} {}_f N \, {}_f\widetilde{e},} $$

[4] It is a cycle in $\widetilde{U-S}$ *with non-compact support.* The appropriate "family of supports" is specified in the appendix (§3).

[5] All the expressions in inverted commas, which seem intuitive enough (?), do not mean very much for the time being. See §2.5 later on.

[6] Cf. [29]. In reality, this formula is only proved in [29] for covering spaces with "independent ramifications" [cf. note 3, p. 130], but I believe that it is true in general. In any case, since it is true for the *universal covering space* of $U-S$, it will also be true for every class $\widetilde{h}$ on any covering space, which is the projection of a class coming from the universal covering space, i.e., for every cycle $\widetilde{\Gamma}$ which is still a cycle after lifting it to the universal covering (when I speak of the class $\widetilde{h}$ of a cycle $\widetilde{\Gamma}$, I naturally mean the trace of this class, or this cycle, in $\widetilde{U-S}$: the problem is purely local).

where

$$(\widetilde{L}) \qquad \boxed{{}_f N = (-)^{(n+1)(n+2)/2} \left\langle {}_f \boldsymbol{e} \,\middle|\, \widetilde{h} \right\rangle .}$$

To see that the sum in $(\widetilde{P})$ makes sense, observe that even if the fibre F were infinite, there would only be a *finite number* of non-zero intersection indices, because $\widetilde{h}$ is a homology class with *compact supports.*

2.4 Application to the ramification of an integral

We can deduce the ramification of an integral $J(t) = \int_{\widetilde{h}} \varphi_t$ from formula $(\widetilde{PL})$, where φ_i is a closed differential form on $\widetilde{X - S}$ which depends holomorphically on t. By defining $\mathrm{Disc}_L\ J(t) = (1 - \omega_L^*)J(t)$ to be the "discontinuity" of $J(t)$, we have

$$(\mathrm{Disc}\,1) \qquad \mathrm{Disc}_L\ J(t) = -\sum_{f \in F} {}_f N \int_{{}_f \widetilde{e}} \varphi_t .$$

If the integrand is "not too singular" in a neighbourhood of $S_1, S_2, \ldots, S_m$ [in practice, if its modulus does not grow faster than $1/|s_i|^a$ $(0 < a < 1)$ as one tends towards S_i in some arbitrary direction; for example, the function $\log s_i/|s_i|^b$ $(0 < b < 1)$], we can replace the integration cycle ${}_f\widetilde{e}$ by ${}_f e = (1 - \omega_1) \cdots (1 - \omega_m)_f \boldsymbol{e}$, which gives

$$(\mathrm{Disc}\,2) \qquad \mathrm{Disc}_L\ J(t) = -\sum_{f \in F} {}_f N \int_{{}_f \boldsymbol{e}} \mathrm{Disc}_1\, \mathrm{Disc}_2 \cdots \mathrm{Disc}_m\ \varphi_t$$

[where we have set $\mathrm{Disc}_i\ \psi = (1 - \omega_i^*)\psi$].

More generally, let us suppose that φ_t behaves as stated above on the manifolds $S_1, S_2, \ldots, S_\mu$, but has (unramified) *polar* singularities on the manifolds $S_{\mu+1}, \ldots, S_m$. In this case, we show that the cycle ${}_f\widetilde{e}$ can be replaced by

$$\delta_{\mu+1} \circ \cdots \circ \delta_m (1 - \omega_1)(1 - \omega_2) \cdots (1 - \omega_\mu)_{*f} \boldsymbol{e}^{\mu+1\ldots m},$$

where δ_i is the Leray coboundary "around" the submanifold S_i, and ${}_f \boldsymbol{e}^{\mu+1\ldots m}$ denotes the cell given by the iterated boundary ${}_f \boldsymbol{e}$:

$${}_f \boldsymbol{e}^{\mu+1\ldots m} = \partial_m \circ \cdots \circ \partial_{\mu+1}\, {}_f \boldsymbol{e}.$$

Leray's residue theorem therefore gives

$$(\mathrm{Disc}\,3) \quad \mathrm{Disc}_L\ J(t) = -(2\pi i)^{m-\mu} \sum_{f \in F} {}_f N \times \int_{{}_f \boldsymbol{e}^{\mu+1\ldots m}} \mathrm{Res}_{\mu+1} \cdots \mathrm{Res}_m\, \mathrm{Disc}_1\, \mathrm{Disc}_2 \cdots \mathrm{Disc}_\mu\ \varphi_t .$$

2.5 Generalization to "relative" homology

If the form φ_t is "not too singular" (cf. §2.4) on $S_1, S_2, \ldots, S_\mu$ ($\mu \leqslant m$), it can even be integrated along chains which meet $S_1, S_2, \ldots, S_\mu$, and it is natural to try to solve the ramification problem of the integral $J(t) = \int_{\widetilde{\Gamma}_{1.2\ldots\mu}} \varphi_t$, where the integration chain $\widetilde{\Gamma}_{1.2\ldots\mu}$ "has boundary" in $S_1 \cup S_2 \cup \cdots \cup S_\mu$. Unfortunately, it is a little delicate to define this idea of taking a chain which has to be "ramified" around S, and "bounding in S". In [29], we find a way out by considering the obvious stratification of Y by the manifolds

$$Y - S, \quad S_i - \bigcup_{k\neq i} S_k, \quad S_i \cap S_j - \bigcup_{k\neq i,j} S_k,$$

and by "attaching together" the covering spaces of each of these strata to form a space $\widetilde{Y}$. We then assume that the integration takes place on a homology class $\widetilde{h}_{1.2\ldots\mu} \in H_*(\widetilde{Y - S}_{\mu+1\ldots m}, \widetilde{S}_{1.2\ldots\mu})$. This procedure has the disadvantage of not being applicable to all possible covering spaces of $Y - S$: one shows that it only gives those for which the ramifications are "independent" [note 3, p. 130], and furthermore, one can only obtain infinite covering spaces at the cost of some unpleasant acrobatics. It is preferable, however, not to try to enlarge the space $\widetilde{Y - S}$: the appendix (§3) shows that we can redefine all the homologies above by choosing appropriate *families of supports* on $\widetilde{Y - S}$, and that the new definition has the advantage of being applicable to all possible covering spaces.

The result of [29] can be stated thus:

$$(\widetilde{P}_{1,2\ldots\mu}) \qquad \boxed{\operatorname{Var} \widetilde{h}_{1,2\ldots\mu} = \sum_{f\in F} {}_f N\, \omega_1\, \omega_2 \cdots \omega_{\mu *}\, {}_f\widetilde{e}_{1,2\ldots\mu},}$$

with

$$(\widetilde{L}_{1,2\ldots\mu}) \qquad \boxed{{}_f N = (-)^\mu (-)^{(n+1)(n+2)/2} \left\langle {}_f\widetilde{e}_{\mu+1\ldots m} \,\middle|\, \widetilde{h}_{1,2\ldots\mu} \right\rangle,}$$

where ${}_f\widetilde{e}_{1.2\ldots\mu}$ is a chain constructed by moving the chain $(1-\omega_{\mu+1})\cdots(1-\omega_m)_*\, {}_f\boldsymbol{e}$ "away from" $S_{\mu+1\ldots m}$, and ${}_f\widetilde{e}_{\mu+1\ldots m}$ is defined in a similar manner. From this formula, we deduce a discontinuity formula which is analogous to (Disc 2), with $\operatorname{Disc}_{\mu+1} \cdots \operatorname{Disc}_m \varphi_t$ as the integrand.

3 Appendix on relative homology and families of supports

3.1 Lemma. *Let $X = Y - S$ be a locally compact and paracompact topological space, such that in Y, S is a deformation retract of some open neighbourhood U, and $g_\tau : U \to U$ is the homotopy between the identity ($g_0 = 1_U$) and* *the retraction r of U onto S ($g_1 = i \circ r$). Let us suppose that the following properties are satisfied:*

(i) (stability of S)*: for every $\tau \in [0,1], g_\tau(S) = S$;*
(ii) (decreasing neighbourhoods)*: for $\tau < 1, g_\tau$ sends U* homeomorphically *onto an open set U_τ, which* decreases *as τ increases. If we call $g_{\tau\tau'} : U_\tau \to U_{\tau'}$ the resulting homeomorphisms, we demand that*

$$\forall \tau'' > \tau' > \tau, \quad g_{\tau\tau'}(U_{\tau''}) \subset U_{\tau''};$$

(iii) for every compact set K of X, we can find $\tau < 1$ such that

$$K \cap U_\tau = \varnothing.$$

Let Φ be a family of supports of X, such that:

(Φ_i) for all $\tau < 1$, $\Phi \cap (Y - U_\tau) =$ the family of compact sets;
(Φ_{ii}) for all $A \in \Phi | U - S$,

$$\bigcup_{\tau<1} g_\tau(A) \in \Phi | U - S.$$

Then we have a canonical isomorphism

$$\boxed{H_*({}_\Phi X) \approx H_*(Y, S).}$$

Examples. The family (U_τ, g_τ) is easy to construct when Y is a differentiable manifold and S is a closed submanifold, by taking U to be a tubular neighbourhood of S as in §I.6.4.

The following are examples of families Φ:

1° $c \cap (Y - S)$, where c is the family of all compact subsets of Y;
2° the family of all the closed subsets of X whose intersection with $Y - U_\tau$ is compact for all $\tau < 1$.

Proof of lemma 3.1. An obvious excision and deformation retract argument gives the canonical isomorphism

$$H_*(Y, S) \approx H_*(X, U - S).$$

On the other hand, by applying condition (Φ_i) with $\tau = 0$, we have an obvious homomorphism

$$\varphi : C_*\left({}_\Phi X\right) \longrightarrow C_*(X, U - S)$$

defined by suppressing the chain elements in every chain of $C_*({}_\Phi X)$ which are supported on $U - S$.

This homomorphism is obviously *surjective*, and its kernel is the group C_* $({}_\Phi | U - S)$. We therefore have an exact sequence[7]

$$0 \longrightarrow C_*\left({}_\Phi | U - S\right) \xrightarrow{j} C_*\left({}_\Phi X\right) \xrightarrow{\varphi} C_*(X, U - S) \longrightarrow 0$$

[7] The rest of the argument will require that the reader is familiar with some notions relating to the homology of chain complexes (*cf.*, for example, [22], chap. II).

which gives a long exact homology sequence

$$\cdots \longrightarrow H_p\left({}_\Phi|U-S\right) \xrightarrow{j_*} H_p\left({}_\Phi X\right) \xrightarrow{\varphi_*} H_p\left(X, U-S\right) \xrightarrow{\partial_*} H_{p-1}\left({}_\Phi|U-S\right) \longrightarrow \cdots$$

We will show that $H_*({}_\Phi|U-S) = 0$, so that φ_* *will be an isomorphism*, and the lemma will be proved.

For every pair $\tau < \tau'$, the map

$$g_{\tau\tau'} : U_\tau - S \longrightarrow U_\tau - S$$

(which is the restriction of $g_{\tau'} \circ g_\tau^{-1}$ to $U_\tau - S$) is homotopic to the identity.

Let $G_{\tau\tau'} : C_p({}_\Phi|U_\tau - S) \to C_{p+1}({}_\Phi|U_\tau - S)$ be the associated homotopy operator: to see that this operator does indeed have the effect suggested by the notation above, one must check that if γ is a locally finite chain with support in $\Phi|U_\tau - S$, then:

1° $G_{\tau\tau'}\gamma$ *is a locally finite chain:* let K be a compact set of $U_\tau - S$. There exists, by (iii), a $U_{\tau''}$ which does not intersect K. Let us set $\gamma = \gamma' + \gamma''$, where γ'' consists of chain elements of γ with support in $U_{\tau''}$. By (ii), $G_{\tau\tau'}\gamma''$ also has support contained in $U_{\tau''}$, so that

$$(\operatorname{supp} G_{\tau\tau'}\gamma) \cap K = (\operatorname{supp} G_{\tau\tau'}\gamma') \cap K.$$

Now, by $(\Phi_i), \gamma'$ is a *finite* chain, and the same is therefore true of $G_{\tau\tau'}\gamma'$: we have therefore verified that K only meets a finite number of elements of the chain $G_{\tau\tau'}\gamma$;

2° supp $G_{\tau\tau'}\gamma \in \Phi|U_\tau - S$: this is obvious by (Φ_{ii}) and axiom (Φ_2) for families of supports (§II.5.1):

Let us show that every cycle $z \in Z_p({}_\Phi|U-S)$ is homologous to zero in $H_p({}_\Phi|U-S)$. We have, by definition of the homotopy operator,

$$z_\tau - g_{\tau\tau'*} z_\tau = \partial G_{\tau\tau'} z_\tau \quad \text{for every cycle} \quad z_\tau \in Z_p\left({}_\Phi|U_\tau - S\right).$$

Let $\tau_0 = 0, \tau_1, \tau_2, \dots$ be an infinite increasing sequence which tends to 1, and consider the chain

$$\gamma = \sum_{i=0,1,2,\dots} G_{\tau_i\tau_{i+1}} z_{\tau_i}, \quad \text{with} \quad z_{\tau_i} = g_{\tau_i *} z.$$

It is a *locally finite* chain, since the family $(U_{\tau_i} - S)_i$ is, by (iii), locally finite in X. Furthermore, its support is in ${}_\Phi|U-S$, by condition (Φ_{ii}). Now, we see immediately that $\partial_\gamma = z$, which completes the proof. □

3.2 The inverse image of a family of supports

Let $f : \widetilde{X} \to X$ be a continuous map of locally compact and paracompact topological spaces, and let Φ be a family of supports in X. Let $f_*^{-1}(\Phi)$ denote the family of subsets of $\widetilde{X}$ whose images belong to Φ and which intersect the inverse image of every compact set in a compact set. It is easy to see that $f_*^{-1}(\Phi)$ is a *family of supports*[8] *on* $\widetilde{X}$ and it is *the largest family of supports* $\widetilde{\Phi}$ *such that the map*

$$f : {}_{\widetilde{\Phi}}\widetilde{X} \longrightarrow {}_{\Phi}X$$

is homologically admissible (§II.5.3).

3.3

With the notations of §3.2, we suppose that f is the restriction of a map $f : \widetilde{Y} \to Y$, where $\widetilde{X} = \widetilde{Y} - \widetilde{S}$, $X = Y - S$, and $S = f(\widetilde{S})$. Suppose that $\widetilde{S}$ and S admit neighbourhoods $\widetilde{U}$ and $U = f(\widetilde{U})$ in $\widetilde{Y}$ and Y, equipped with homotopies $\widetilde{g}_\tau$ and g_τ which satisfy the conditions of lemma 3.1, and are compatible with the projection f. Then, if Φ is a family of supports of X, which satisfies the conditions of lemma 3.1, the same is true, as one easily verifies, for the family $f_*^{-1}(\Phi)$ in $\widetilde{X}$. We therefore have a canonical isomorphism

$$H_*\big({}_{f_*^{-1}(\Phi)}\widetilde{X}\big) \approx H_*\big(\widetilde{Y}, \widetilde{S}\big).$$

Application to the problem of §2.5

Given the pair (Y, S) and the covering $r : \widetilde{Y - S} \to Y - S$, we can define the "relative" homology group $H_*(\widetilde{Y}, \widetilde{S})$ without having to construct the space $\widetilde{Y}$ or the space $\widetilde{S}$: we choose a family Φ in $Y - S$ satisfying the conditions of lemma 3.1, and we set, *by definition*,

$$H_*\big(\widetilde{Y}, \widetilde{S}\big) = H_*\big({}_{r_*^{-1}(\Phi)}\widetilde{Y - S}\big).$$

[8] This family of supports is cited by Serre in the *Séminaire Cartan* 1950–1951 (Exposé 21) [4], to illustrate the notion of "well-adapted families".

Technical notes

> "Analyste! rends hommage à la Vérité, sinon l'équidomoïde (12) vengeur viendra peser, la nuit, sur ta poitrine anxieuse."
>
> (Léopold Hugo)

(1) A topological space is called "*Hausdorff*" if any two distinct points always have two neighbourhoods which are disjoint. The fact that a space is locally homeomorphic to $\mathbb{R}^n$ [property (X_0) of chapter I, §1] does not imply that it is Hausdorff: an important counter-example is the "sheaf of germs of continuous functions on $\mathbb{R}^n$".

(2) P^n is equipped with the "quotient topology" of the topology on $\mathbb{R}^{n+1}-\{0\}$ by the equivalence relation being considered. In other words, the open sets of P^n are the images of the open sets of $\mathbb{R}^{n+1}-\{0\}$.

(3) A *Hausdorff* topological space X is called "*paracompact*" if *every open cover of X has a locally finite refinement*. One proves that such a space is "*normal*", i.e., that two disjoint closed sets always have two neighbourhoods which are disjoint. A useful property of normal spaces is the following: for every open covering $\{U_i\}$ of X, there exists an open covering $\{V_i\}$ such that $\overline{V}_i \subset U_i$.
The connection between paracompactness of manifolds and property (X_1) of chapter I (§1) is clear, due to the following theorem:
For a *locally compact* Hausdorff space to be *paracompact*, it is necessary and sufficient that each of its connected components be a *countable* union of compact sets.

(4) The theorem of the *topological invariance of open sets* states that if two subspaces of Euclidean space $\mathbb{R}^n$ are *homeomorphic*, and if one is *open* (in $\mathbb{R}^n$), the other is too. For a proof of this subtle theorem, cf., for example, ([8], chap. XI).

(5) For completeness, I should mention two further properties of homology:

(i) *Excision: Let (X, A) be a pair, and let U be a subset of A whose closure is contained in the interior of A. There is a canonical isomorphism*

$$H_*(X, A) = H_*(X - U, A - U).$$

This property, which is not obvious to prove, is intuitively quite clear: since the relative homology of the pair (X, A) "*ignores*" the chains of A, it is not surprising that we can remove a piece of A.

(ii) *Exact homology sequence*: the sequence

$$\cdots \xleftarrow{\partial_*} H_p(X, A) \leftarrow H_p(X) \xleftarrow{i_*} H_p(A) \xleftarrow{\partial_*} H_{p+1}(X, A) \leftarrow \cdots$$

(where the unnamed arrow represents the homomorphism induced by the projection map $C_*(X) \to C_*(X, A)$) *is exact*, i.e., the image of each homomorphism is equal to the kernel of the next homomorphism.

(6) It remains to mention two properties of cohomology.

(i) *Excision*: can be expressed exactly as for homology (5).

(ii) *Exact cohomology sequence*: this is the sequence

$$\cdots \xrightarrow{\delta^*} H^p(X, A) \to H^p(X) \xrightarrow{i^*} H^p(A) \xrightarrow{\delta^*} H^{p+1}(X, A) \to \cdots$$

[where the unnamed arrow represents the homomorphism induced by the inclusion map $C^*(X, A) \to C^*(X)$].

(7) In general, we have the *exact cohomology sequence of a closed subspace S*

$$\cdots \xrightarrow{\delta^*} H^p({}^{\Phi}|X - S) \to H^p({}^{\Phi}X) \xrightarrow{i^*} H^p({}^{\Phi}|S) \xrightarrow{\delta^*} H^{p+1}({}^{\Phi}|X - S) \to \cdots$$

If we are dealing with oriented manifolds, Poincaré's isomorphism transforms this sequence into *Leray's exact homology sequence*, which in the special case $\Phi = c$, can be simply written

$$\cdots \xrightarrow{\delta^*} H_q(X - S) \xrightarrow{j_*} H_q(X) \xrightarrow{i^*} H_{q-r}(S) \xrightarrow{\delta^*} H_{q-1}(X - S) \xrightarrow{j_*} \cdots$$

$$(q = \dim X - p,\ r = \text{codim } S),$$

where j_* is the homomorphism induced by the inclusion $j : X - S \to X$, and i^* can be interpreted as the "*intersection*" of cycles of X with S.

(8) We often introduce an extra structure in a locally trivial fibre bundle by specifying a *group G of automorphisms of the fibre* and by requiring that the various local trivializations can be "patched together" with transformations which belong to this group (for example, in the case of the tangent bundle of a manifold, G could be the linear group). We then say that we have a "*fibre bundle with a structure group*".

(9) Since the formulation of Whitney's conditions A and B involves the "angles" between hyperplanes, one might expect that the notion of regular incidence brings into play the *metric* structure of the ambient space. This

is not at all the case, and the conditions that certain angles between hyperplanes tend to zero can be rephrased by saying that certain points of a "fibre bundle whose fibres are Grassmannians" [3] have a limit in certain "Schubert cycles" [3] of this space (cf. [38]; in fact, [38] also involves the "distance" in the Grassmannian, but one can get around this by speaking of the "uniform topology" in the Grassmannian).

(10) Let Y be a differentiable manifold, and let M and N be two differentiable submanifolds of Y, of codimension m' and n' respectively. We say that M and N are *transversal* at the point $y \in M \cap N$ if their tangent spaces at y span the whole tangent space $T_y(Y)$:

$$T_y(M) + T_y(N) = T_y(Y).$$

(This coincides with the notion of "general position" in codimension 1.) M and N are *transversal* if they are transversal at every point where they intersect. $M \cap N$ is then a submanifold of codimension $m' + n'$ (with the convention that a manifold of negative dimension is empty).
A differentiable map $f : X \to Y$ is said to be *transversal* to a submanifold $M \subset Y$ if its graph is transversal to the submanifold $X \times M \subset X \times Y$. It is said to be *transversal to a stratified set* $L \subset Y$ if it is transversal to each of the strata of L (observe that if f is transversal to a stratum of L, then by Whitney's condition A, it will also be transversal to its star near this stratum).

(11) One question immediately springs to mind: given a closed form ω on S, can one find a closed form φ on $X - S$ whose residue is ω? The answer to this question is given by the exact cohomology sequence

$$\cdots \xleftarrow{\text{Res}} H^q(X-S) \xleftarrow{j^*} H^q(X) \xleftarrow{i^{*T}} H^{q-2}(S) \xleftarrow{\text{Res}} H^{q-1}(X-S) \xleftarrow{j^*} \cdots$$

which is the transpose of Leray's exact homology sequence [cf. note (7), with $r = 2$] with respect to the bilinear form given by "integration" (de Rham duality).
We can see from this sequence that a necessary and sufficient condition for a cohomology class $h^{q-2} \in H^{q-2}(S)$ to be the residue of a class in $H^{q-1}(X - S)$ is that the image of h^{q-2} under the homomorphism i^{*T} is zero (for an explicit construction of i^{*T}, cf. Leray [20]). In particular, this is always the case when S and X are smooth algebraic subvarieties of $\mathbb{C}^N$ which intersect the "hyperplane at infinity" in $\mathbb{C}P^N$ (the obvious compactification $\mathbb{C}^N$) transversally (cf. [10], the "decomposition theorem", where it is shown that under such conditions, the homomorphism $i^* : H_q(X) \to H_{q-2}\,(S)$ is zero; the same is therefore true for its transpose i^{*T}).

(12) For the definition of the word "équidomoïde", cf. [31], which is where I found this citation.

Sources

Here is a rough list of the origin of the main ideas which feature in this text – apart from possible errors, which are my own contribution.

I. For generalities on *differentiable manifolds*, cf. Whitney [40], or de Rham [6]. I was also inspired by a course given by Deheuvels at the Institut Henri Poincaré (1962–1963).

II. A modern – and very elementary – account of *homology* can be found in Wallace [39]. Cf. also Eilenberg and Steenrod [8], to whom the *axiomatic* presentation of the theory is due, and Mac Lane [22], who gives an account of *algebraic* techniques.

The notion of a *family of supports*, as well as the general form of *Poincaré's isomorphism* are due to H. Cartan [4]. Some aspects of this are summarized in [12].

The notion of a *current* was invented by de Rham [6]; cf. also [28].

III. Leray [20]. Cf. also Norguet's talk [26] for a more general formulation [27] of *Leray's theory of residues*.

IV. The notion of *ambient isotopy* is common in the literature, but the terminology varies quite widely. I have stuck to that of [12].

The notion of a *stratified set* is due to Whitney [42] and Thom [37]. My version of Whitney's *regular incidence* conditions is taken from [38].

The first time Thom's *isotopy theorem* was stated, along with a sketched proof, was in [37]. A detailed proof was presented in the Séminaire Thom–Malgrange on differential geometry (I.H.E.S., Bures-sur-Yvette, 1964–1965), and a write-up is in preparation. In fact, to prove this theorem, Thom uses a slightly different definition of regular incidence from Whitney's, which essentially consists of a "patching" property (the existence of functions which "patch together" to form the boundaries of the strata). It seems, however, that this patching property is a consequence of Whitney's conditions A and B (Thom has recently announced this in a letter to D. Fotiadi).

There are two remarks to make about the statement of the theorem.

1° Thom takes the segment $[0, 1]$ as his base space T, but it is easy to generalize to the case of a q-dimensional cube (by induction on q: cf. [12]), which gives local triviality over any manifold of dimension q.
2° A literal reading of the statement of Thom's theorem in [37] might lead one to think that the *surjectivity* of the map π restricted to each stratum should be added to the assumptions. In reality, it can be deduced from the other hypotheses (namely, that π is proper of rank $(\pi|A) = \dim T$, and that the strata satisfy the regular incidence property).
For *singularities of differentiable maps*, cf. Whitney [41] and Thom [36].

V. For the ramification of homology classes (*the Picard–Lefschetz formula*), cf. Lefschetz [19]. A more explicit proof is given in Fary [10]. These authors are only interested in the homology $H_n(S)$ for a single submanifold S [which corresponds to formula (PL) of §§V.2.4, V.2.5].
The case of the homology group $H_n(X - S)$ where S is a *union* of submanifolds in general position can be deduced from this by a generalized version of Poincaré duality: cf. Fotiadi, Froissart, Lascoux and Pham [12]. An independent treatment of this is given in my article [29].

VI. This whole chapter is essentially copied from Leray's article [20], except that I consider a *union* of submanifolds. This is an easy generalization, albeit a little fastidious.

VII. Cf. my article [29].
There are some generalities on *fibre bundles*, *covering spaces*, *homotopy groups* etc, spread over different chapters. For all these notions, cf. the Séminaire Henri Cartan [3] and the books of Steenrod [34] and Hilton [16], etc.

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Part II

Introduction to the study of singular integrals and hyperfunctions

Introduction

We shall consider a class of distributions that one meets in many problems, and in particular in mathematical physics. The study of these distributions raises many interesting geometric problems – namely, those related to the monodromy of singularities.

Let us consider some examples in the one variable case.

First, we recall the following well-known formulae which are used by physicists:

$$\frac{1}{x \pm i0} = P\left(\frac{1}{x}\right) \mp i\,\pi\,\delta(x)$$
$$\frac{1}{(x \pm i0)^{n+1}} = P\left(\frac{1}{x^{n+1}}\right) \mp \frac{(-1)^n}{n!} i\,\pi\,\delta^{(n)}(x).$$

From these we obtain the following expressions for the distributions $P\left(1/x^{n+1}\right)$ and $\delta^{(n)}(x)$:

$$P\left(\frac{1}{x^{n+1}}\right) = \frac{1}{2}\left\{\frac{1}{(x+i0)^{n+1}} + \frac{1}{(x-i0)^{n+1}}\right\}$$
$$\delta^{(n)}(x) = \frac{n!}{2\pi i}(-1)^{n+1}\left\{\frac{1}{(x+i0)^{n+1}} - \frac{1}{(x-i0)^{n+1}}\right\}.$$

In these formulae, $1/(x+i0)^{n+1}$ (resp. $1/(x+i0)^{n+1}$) is the boundary value of the analytic function $1/z^{n+1}$ as one approaches the real axis from above (resp. from below).

Let us consider another example: the Heaviside distribution, which is defined by the function

$$Y(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ 1 & \text{for } x > 0. \end{cases}$$

In other words, the distribution can be written

$$\frac{1}{2\pi i}\big(\log(x-i0) - \log(x+i0)\big),$$

where $\log(x+i0)$ and $\log(x-i0)$ are boundary values of the analytic function $\log z$ on both sides of the real axis.

More generally, every distribution can be written as a sum of boundary values of analytic functions.

We will limit ourselves to studying a particular class of analytic functions which will give us a particular class of distributions.

VIII

Functions of a complex variable in the Nilsson class

In the case of one variable, prototypes for the functions that we will constantly be using are:

$$\begin{aligned} &z^\alpha, &&\alpha \in \mathbb{C} \\ &\log^p z, &&p \in \mathbb{N}. \end{aligned}$$

What is remarkable about these functions?

1) They are multivalued analytic functions on $\mathbb{C} - \{0\}$.
2) They are functions of finite determination, i.e., the branches of each function span a vector space V (over $\mathbb{C}$) of finite dimension μ. In fact, by analytic continuation along a loop which winds once around the origin in the positive direction, the branches of these functions become:

$$\begin{aligned} z^\alpha &\longmapsto \mathrm{e}^{2\pi i\,\alpha} z^\alpha \qquad \text{(and thus} \quad \mu = 1) \\ \log^p z &\longmapsto (\log z + 2\pi i)^p \end{aligned}$$

which can be written as a linear combination of the functions $1, \log z$, $\log^2 z, \ldots, \log^p z$, where $\mu = p + 1$.

1 Functions in the Nilsson class

More generally, one can define the notion of a function of finite determination on a complex analytic manifold as follows:

1.1 Definition. Let X be a complex analytic manifold, and let f be a multivalued analytic function on X (i.e., a holomorphic function on a covering $\widetilde{X}$ of X; note that $\widetilde{X}$ is not necessarily simply-connected). We say that f is of *finite determination* on X if its set of branches at every point of X (or rather, on every simply connected open subset of X) spans a finite-dimensional vector space over $\mathbb{C}$. These vector spaces form a locally constant sheaf $\mathscr{V}$ of finite-dimensional vector spaces on X which we will call the *sheaf of determinations* of f.

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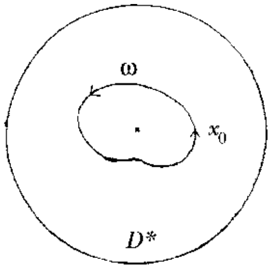
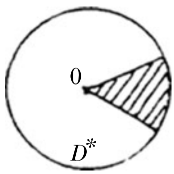
Let f be a multivalued function of finite determination on X, and let $\mathscr{V}$ be the sheaf of determinations of f. The analytic continuation of a branch $g \in \mathscr{V}_{x_0}$ along a loop starting at x_0 will in general give a different determination $g' \in \mathscr{V}_{x_0}$. In this way, we obtain an invertible linear map from $\mathscr{V}_{x_0}$ to itself which only depends on the homotopy class of the loop. In other words, if $\pi_1(X, x_0)$ denotes the fundamental group of homotopy classes, then we have a group homomorphism: $\pi_1(X, x_0) \to \mathrm{GL}(\mathscr{V}_{x_0})$, which we shall call the *monodromy* of f (or of $\mathscr{V}$). Every transformation $\mathscr{T} \in \mathrm{GL}(\mathscr{V}_{x_0})$ in the image of this homomorphism is called a *monodromy transformation*. The matrix of this transformation, in a fixed basis of $\mathscr{V}_{x_0}$, is called a *monodromy matrix*.

Recall that in the case $X = \mathbb{C} - \{0\}$, or $X = D^*$ (the punctured disk), $\pi_1(X, x_0) \simeq \mathbb{Z}$ is spanned by the homotopy class of a loop ω at x_0 which winds once around the origin in the positive direction. The monodromy transformation of f corresponding to ω is again called the monodromy transformation of f and its matrix in a basis of $\mathscr{V}_{x_0}$ is called the monodromy matrix of f.



1.2 Definition. Let f be a multivalued analytic function of finite determination on the punctured disk D^*. We say that f has *moderate growth* near the origin if every branch of f in an angular sector has an upper bound of the form

$$|f(z)| \leqslant \frac{C}{|z|^a},$$



where C and a are real constants, and $C > 0$.

Note that C may depend on the branch of f. (For example, the function $f = \log z$ does not have an upper bound of the form given above with the same constant C for all branches.)

Example. The functions z^α, $\log^p z$ considered earlier have moderate growth near the origin.

1.3 Theorem. *Let* $D = \{z \in \mathbb{C}; |z| < 1\}$, *and* $D^* = \{z \in \mathbb{C}; 0 < |z| < 1\}$.

Let f be a multivalued analytic function on D^*. Then f is of finite determination (resp. of finite determination and of moderate growth near the origin) if and only if f can be written in the form

$$f(z) = \sum_{i \in I} z^{\alpha_i} \, P_i(\log z), \tag{1.4}$$

where I is a finite set of indices, $\alpha_i \in \mathbb{C}$, and $P_i(w)$ is a polynomial in w with coefficients in $\mathscr{O}(D^*)$ (resp. in $\mathscr{O}(D)$) (here we use the notation $\mathscr{O}(X)$ to denote the set of holomorphic functions on a complex analytic manifold X).

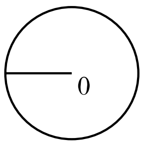
Proof. Observe first of all that by taking a Laurent (resp. Taylor) expansion of the coefficients, we find that f is of the form (1.4) if and only if it can be written in the form

$$f(z) = \sum_{k \in \mathbb{Z}} \sum_{j \in J} c_{j,k} \, z^{\alpha_j + k} \, \log^{p_j} z \tag{1.5}$$

where J is a finite set of indices, $\alpha_j \in \mathbb{C}$, $p_j \in \mathbb{N}$, and $c_{j,k} \in \mathbb{C}$ (resp. f can be written in the form (1.5) with the index k running over $\mathbb{N}$).

In this form, the condition is clearly sufficient. To prove that the condition is necessary, we first consider the case $\mu = 1$. The proof in the general case is based on the same idea.

Therefore, let us assume that the vector space of determinations of f in $D - \mathbb{R}^-$ is spanned by a single generator f_0. In the basis $\{f_0\}$, the monodromy transformation is defined by $f_0 \mapsto cf_0$, where $c \in \mathbb{C}$. Therefore, there exists an $\alpha \in \mathbb{C}$ such that $c = \mathrm{e}^{2\pi i \alpha}$.



$D - \mathbb{R}^-$

By analytic continuation along a loop which defines the monodromy, the function z^α becomes $\mathrm{e}^{2\pi i \alpha} z^\alpha$, and as a result, the function $f_0 z^{-\alpha}$ remains invariant after this analytic continuation. In other words, $f_0 z^{-\alpha}$ is a single-valued analytic function on D. This gives $f = \varphi z^\alpha$, where $\varphi \in \mathscr{O}(D^*)$, which is indeed of the form (1.4).

Suppose, furthermore, that f has moderate growth. Then $z^{-\alpha} f$ is a single-valued analytic function on D and is bounded above by a power of z, and is therefore meromorphic, by a theorem due to Liouville. In other words, $f(z) = z^{\alpha - n} g(z)$ with $n \in \mathbb{Z}$ and $g \in \mathscr{O}(D)$.

Now let us consider the general case. Let $b = (b_1, \ldots, b_\mu)$ be a system of multivalued analytic functions on D^* which form a basis of the vector space spanned by the branches of f on $D - \mathbb{R}^-$ (or, rather, a basis of the vector space of determinations $\mathscr{V}_{\widetilde{D}^*}$ on a covering $\widetilde{D}^*$ of D^*). Let $\mathscr{T}$ (resp. $\mathscr{M}$) be the monodromy transformation (resp. the monodromy matrix in the basis b) which corresponds to a generator ω of $\pi_1(D^*, x_0)$. Then we have a commutative diagram:

$$\begin{tikzcd} \mathbb{C}^\mu \arrow[r, "b"] \arrow[d, "\mathscr{M}"'] & \mathscr{V}_{\widetilde{D}^*} \arrow[d, "\mathscr{T}"] \\ \mathbb{C}^\mu \arrow[r, "b"] & \mathscr{V}_{\widetilde{D}^*} \end{tikzcd}$$

In other words, $\omega(b) = \mathscr{T} \circ b = b \circ \mathscr{M}$, where $\omega(b) = (\omega(b_1), \dots, \omega(b_\mu))$ is the analytic continuation of b along the loop ω.

Since $\mathscr{M}$ is invertible, there exists a complex matrix A such that $\mathscr{M} = \mathrm{e}^{2\pi i \mathrm{A}}$. Let us consider the matrix-valued function $z \to z^{\mathrm{A}} = \mathrm{e}^{\mathrm{A} \log z}$. By doing an analytic continuation of the coefficients of this matrix along the loop ω, we obtain

$$\omega\left(z^{\mathrm{A}}\right) = \mathrm{e}^{\mathrm{A}(\log z + 2\pi i)} = z^{\mathrm{A}} \mathscr{M}.$$

Now consider the automorphism $z^{-\mathrm{A}}$ of the sheaf $\mathscr{O}^\mu_{\widetilde{D}^*}$ and the homomorphism β from the sheaf $\mathscr{O}^\mu_{\widetilde{D}^*}$ to the sheaf $\mathscr{O}_{\widetilde{D}^*}$ defined by

$$\beta\left(g_1, \dots, g_\mu\right) = \sum_{i=1}^{\mu} g_i b_i, \quad (g_1, \dots, g_\mu) \in \mathscr{O}^\mu_{\widetilde{D}^*}.$$

Let us set $\gamma = \beta \circ z^{-\mathrm{A}}$. Then we observe that the restriction of γ to $\mathscr{O}^\mu_{D^*}$ gives a homomorphism of sheaves: $\mathscr{O}^\mu_{D^*} \to \mathscr{O}_{D^*}$. To see this, let $g = (g_1, \dots, g_\mu) \in \mathscr{O}^\mu_{D^*}$. We have:

$$\begin{aligned} \omega(\gamma(g)) &= \omega(\beta) \circ \omega\left(z^{-\mathrm{A}}\right)(\omega(g)) \\ &= \beta \circ \mathscr{M} \circ \mathscr{M}^{-1} \circ z^{-\mathrm{A}}(g) \\ &= \beta \circ z^{-\mathrm{A}}(g) = \gamma(g). \end{aligned}$$

This shows that $\gamma(g)$ is a single-valued function on D^*, i.e., $\gamma(g) \in \mathscr{O}_{D^*}$. In particular, the functions $g_j = \gamma(e_i)$ are single-valued, where $\{e_i\}$ is the canonical basis of $\mathbb{C}^\mu$. Here, $\mathbb{C}^\mu$ is embedded in $\mathscr{O}_{D^*}$ in a natural way. Now we obtain condition (1.4) by writing out the formula $\mathscr{V}_{\widetilde{D}^*} = \beta(\mathbb{C}^\mu) = \gamma \circ z^{\mathrm{A}}(\mathbb{C}^\mu)$ explicitly.

One can put A into Jordan canonical form by choosing another basis of $\mathbb{C}^\mu$:

$$\mathrm{A} = \left[\begin{array}{cccc:c:cccc}
\lambda & 1 & & \bigcirc & & & & & \\
 & \lambda & & & & & 0 & & \\
 & & \ddots & 1 & & & & & \\
\bigcirc & & & \lambda & & & & & \\
\hdashline
 & & & & \ddots & & & & \\
\hdashline
 & & & & & \nu & 1 & & \bigcirc \\
 & 0 & & & & & \nu & & \\
 & & & & & & & \ddots & 1 \\
 & & & & & \bigcirc & & & \nu
\end{array}\right]$$

We set

$$\mathrm{S} = \begin{bmatrix}
\lambda & & & & & \\
 & \ddots & & & 0 & \\
 & & \lambda & & & \\
 & & & \ddots & & \\
 & & & & \mu & \\
 & 0 & & & & \ddots \\
 & & & & & & \mu
\end{bmatrix}$$

and

$$\mathrm{N} = \left[\begin{array}{cccc:c:cccc}
0 & 1 & & \bigcirc & & & & & \\
 & 0 & & & & & 0 & & \\
 & & \ddots & 1 & & & & & \\
\bigcirc & & & 0 & & & & & \\
\hdashline
 & & & & \ddots & & & & \\
\hdashline
 & & & & & 0 & 1 & & \bigcirc \\
 & 0 & & & & & 0 & & \\
 & & & & & & & \ddots & 1 \\
 & & & & & \bigcirc & & & 0
\end{array}\right]$$

Then $\mathrm{A} = \mathrm{S} + \mathrm{N}$, where N is a nilpotent matrix, i.e., $\mathrm{N}^m = 0$ for some $m \in \mathbb{N}$, and N commutes with S. We therefore have

$$z^{\mathrm{A}} = z^{\mathrm{S}} z^{\mathrm{N}} = z^{\mathrm{S}} \mathrm{e}^{\mathrm{N} \log z} = z^{\mathrm{S}} \left(1 + \mathrm{N} \log z + \cdots + \frac{n^{m-1} \log^{m-1} z}{(m-1)!} \right) \cdot$$

It is clear that the coefficients of z^{A}, and as a result, the components of every vector $z^{\mathrm{A}}(v)$, $v \in \mathbb{C}^\mu$, are of the form

$$\big(z^{\mathrm{A}}(v)\big)_i = z^{\alpha_i}\, Q_i(\log z),$$

where α_i *is an eigenvalue* of A, and Q_i is a polynomial with constant coefficients.

Finally, we obtain:

$$\gamma \circ z^{\mathrm{A}}(v) = \sum g_i(z) z^{\alpha_i}\, Q_i(\log z),$$

which is indeed of the form (1.4), where the $g_i \in \mathscr{O}(D^*)$ are the components of γ defined earlier.

It now remains to consider the case where f has moderate growth near the origin. Then the functions b_i also have moderate growth. Since the coefficients of $z^{-\mathrm{A}}$ obviously have moderate growth, it follows that the components g_i of $\gamma = \beta \circ z^{-\mathrm{A}}$ have moderate growth near the origin. In other words, the g_i are meromorphic functions. Then we can assume that they are holomorphic at 0 by subtracting positive integers from α_i. $\square$

Remark. Let U be an open set of $\mathbb{C}$, and let Y be a subset of isolated points of U. In an analogous way, one can define the notion of a multivalued analytic function on $U - Y$ of finite determination and moderate growth near Y. Such a function is called a *function in the Nilsson class* on U. (Fuchs had already studied these functions, but Nilsson generalized this notion to complex analytic manifolds.)

2 Differential equations with regular singular points

Let U be an open connected subset of $\mathbb{C}$. Consider the equation

$$\frac{d^\mu}{dz^\mu} f + a_1(z) \frac{d^{\mu-1}}{sz^{\mu-1}} f + \cdots + a_\mu(z) f = 0. \tag{2.1}$$

(i) If all the a_i are holomorphic on U, it is well known that the set of local solutions of (2.1) forms a locally constant sheaf of vector spaces of dimension μ.

(ii) On the other hand, if the a_i are meromorphic on U, a theorem of Fuchs states that a necessary and sufficient condition for all the solutions of (2.1) to be functions having moderate growth in the neighbourhood of a pole of the a_i, is that this pole is a *regular singular point* of (2.1), i.e., the order of the pole of a_i is at most i.

Conversely, we have the following:

2.2 Proposition. *Let U be a connected open subset of $\mathbb{C}$.*

(i) If $\mathscr{V} \subset \mathscr{O}_U$ is a locally constant sheaf of vector spaces of dimension μ, then there exists one and only one differential equation of the form (2.1), where the a_i are meromorphic on U, such that $\mathscr{V}$ coincides with the sheaf of local solutions of (2.1) outside the poles of a_i.

(ii) If $\mathscr{V} \subset \mathscr{O}_{U^*}$, where $U^* = U - Y$ and Y is a set of isolated points of U, and if, furthermore, all the sections of $\mathscr{V}$ have moderate growth in a neighbourhood of the points in Y, then $\mathscr{V}$ is the sheaf of solutions of one and only one equation of the form (2.1) with a_i meromorphic on U, and the points in Y are regular singular points of (2.1).

Remarks.

1) The poles of a_i in (i) correspond to the zeros of sections of $\mathscr{V}$. The following example shows that such poles do exist in general:

Let f be a function which is holomorphic on a simply connected open subset U of $\mathbb{C}$, such that f has a zero of order k at z_0:

$$f(z) = (z - z_0)^k u(z), \quad u(z_0) \neq 0.$$

Then f is a solution of the first order differential equation:

$$\frac{df}{dz} + a_1(z) f(z) = 0, \quad \text{with } a_1 = -\frac{k}{z - z_0} - \frac{u'(z)}{u(z)}.$$

We see that a_1 does indeed have a pole at z_0.

2) In (ii), the "moderate growth" hypothesis is indispensable for the poles of a_i outside Y (which will exist in general by remark 1) not to accumulate at points of Y (without which the a_i would not be meromorphic on U).

Proof of proposition 2.2. The uniqueness is obvious by the remarks which precede the proposition. Now let us prove the existence of the differential equation of the form (2.1) which corresponds to the sheaf $\mathscr{V}$.

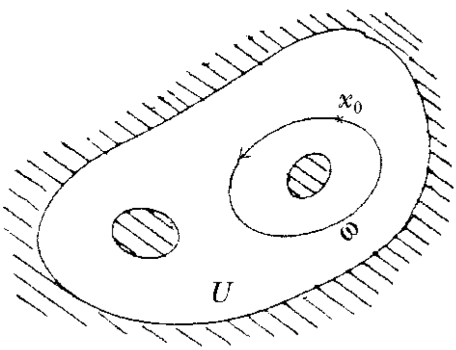
(i) Let $(b_1, \ldots, b_\mu)$ be a basis of $\mathscr{V}$ on a covering $\widetilde{U}$ of U, and let $W(b_1, \ldots, b_\mu)$ be the Wronskian of the b_i, in other words,

$$W(b_1, \ldots, b_\mu) = \operatorname{Det} \begin{bmatrix} b_1 & \ldots & b_\mu \\ b_1' & \ldots & b_\mu' \\ \ldots & \ldots & \ldots \\ b_1^{(\mu-1)} & \ldots & b_\mu^{(\mu-1)} \end{bmatrix}.$$

Since the b_i are linearly independent, $W(b_1, \ldots, b_\mu)$ is not identically zero on $\widetilde{U}$. On the other hand, if f is an arbitrary section of $\mathscr{V}$, then f depends linearly on $b_1, \ldots, b_\mu$, and therefore the Wronskian $W(f, b_1, \ldots, b_\mu)$ is identically zero on $\widetilde{U}$. In other words, f is a solution of the equation

$$\frac{W(f, b_1, \ldots, b_\mu)}{W(b_1, \ldots, b_\mu)} = 0$$

which is indeed of the form (2.1). The coefficients a_i of this equation are single-valued functions on U.



Under the action of an element $[\omega] \in \pi_1(U, x_0)$, every row of the matrix

$$\begin{bmatrix} b_1 & \dots & b_\mu \\ b_1' & \dots & b_\mu' \\ \dots & \dots & \dots \\ b_1^{(\mu)} & \dots & b_\mu^{(\mu)} \end{bmatrix}$$

is multiplied by a matrix with constant coefficients $\mathscr{M}$ (the monodromy matrix corresponding to $[\omega]$), and so the minors of order μ of this matrix are multiplied by Det $\mathscr{M}$. It follows that the a_i, which are nothing other than the quotients of these minors, are invariant with respect to $\pi_1(U, x_0)$, i.e., they are single-valued functions on U. The possible poles of the a_i come from the zeros of $W(b_1, \dots, b_\mu)$.

(ii) Now suppose that the b_i have moderate growth near Y. Then the derivatives of b_i are also of moderate growth near Y (this is an immediate consequence of expression (1.5) in theorem 1.3), and so $W(z) = W(b_1, \dots, b_\mu)$ is a function of moderate growth. From what we have just said, if ω is a small loop which winds around an isolated point $z_0 \in Y$, then $W(z)$ is multiplied, under the action of $[\omega]$, by the determinant of the monodromy matrix defined by ω. This shows, as usual, that in a neighbourhood of z_0, we can write $W(z) = (z - z_0)^\alpha u(z)$, where $\alpha \in \mathbb{C}$ and u is a holomorphic function of z_0 such that $u(z_0) \neq 0$. Now it is clear that the zeros of $W(z)$, and as a result, the poles of the a_i, do not accumulate at z_0. $\square$

IX

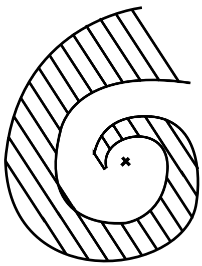
Functions in the Nilsson class on a complex analytic manifold

1 Definition of functions in the Nilsson class

The definition given in chapter VIII, in the case of a single variable, can be generalized as follows.

1.1 Definition. Let X be a complex analytic manifold. We say that f is in the *Nilsson class* on X, and we write $f \in \mathrm{Nils}(X)$, if f is a multivalued analytic function of *finite determination* on $X - Y$, where Y is a complex analytic hypersurface in X, and has *moderate growth* along Y.

To understand this "definition", one needs to know what "moderate growth" means. The difficulty here is to generalize the notion of "angular sector", which we used to define moderate growth in the case of a single variable. The first naïve generalization of an angular sector on a simply connected open subset of $X - Y$ which is relatively compact in X, turns out to be inadequate, even in the case of one variable. In fact, there are simply connected, relatively compact strips in $\mathbb{C}$ which "spiral" around the origin in such a way that the functions with moderate growth near the origin in the usual sense (for example, the function $\log z$) are not bounded above by any power of z in these strips.



The definition that we give below is due to P. Deligne ("Équations différentielles à points singuliers réguliers", Chapter II, §2).

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1.2 Definition. Let $X^* = X - Y$, as previously.

(i) A *single-valued* function f, defined on any subset P^* of X^*, is said to have *moderate growth along* Y if, for every sufficiently small open subset U of X such that $Y \cap U$ is defined by an equation of the form $g(x) = 0$, where g is an analytic function on U, there exist two real numbers N and C, $C > 0$, such that

$$|f(x)| < \frac{C}{|g(x)|^N} \quad \text{on} \quad P^* \cap U. \tag{1.3}$$

(ii) Now let f be a multivalued analytic function on X. Then we say that f has *moderate growth along* Y if every branch of f has moderate growth along Y in the sense of (i) on every simply connected subset of the form $P^* = P - Y$, where P is a *compact semi-analytic* subset of X.

Remarks. 1) To prove that a single-valued analytic function has moderate growth on the set P^*, it suffices to check condition (1.3) on the open sets U_i of an arbitrary but fixed open covering of P^*. Furthermore, for each U_i, it suffices to check it for a single defining equation $g_i(x) = 0$ in $Y \cap U_i$ since, by Hilbert's Nullstellensatz, given two such functions g_i and g'_i, there exists $m' \in \mathbb{N}$ (resp. $m \in \mathbb{N}$) and an analytic function $a'(x)$ (resp. $a(x)$) on U_i such that we have $g_i^{m'}(x) = a'(x)g'_i(x)$ (resp. $g_i'^m(x) = a(x)g_i(x)$). Therefore, an upper bound by a power of g_i still gives an upper bound by a power of g'_i, and vice versa.

2) In the second part of definition 1.2, it suffices to check that each branch of f has moderate growth on $P^* = P - Y$ where P runs over the simplices of a fixed semi-analytic triangulation of X. In fact, by a theorem due to Łojasiewicz, each member of a locally finite family of semi-analytic subsets of X is contained in the union of a finite number of simplices in such a triangulation. Let us consider, for example, the case $X = D$, $Y = \{0\}$. By triangulating X with a finite number of analytic sectors, we retrieve the notion of functions of moderate growth near the origin previously defined in the case of one variable.

3) Now suppose that X^* can be embedded in another complex analytic manifold X' such that $X^* = X' - Y'$, where Y' is a hypersurface of X', and suppose furthermore that there exists a proper analytic map from X' to X such that the image of Y' is Y. We therefore have two notions of functions having moderate growth on X^*: one along Y in X, and the other along Y' in X'. In fact, these two notions coincide. Let us try to see this by applying definition 1.2.

Since the image (resp. the inverse image) of a triangulation of X' (resp. of X) under a proper map is still a triangulation of X (resp. of X'), it would suffice to prove that the image of a semi-analytic set is still semi-analytic – but this is false, alas! Fortunately, Hironaka has introduced the notion of *subanalytic set* which generalizes the notion of a semi-analytic set, and for which everything we have said here remains true (including Łojasiewicz's theorem) with the added bonus that the image of a sub-analytic set under a proper analytic map is still sub-analytic. By an analogous argument to 2), we see that our definition of moderate growth remains unchanged if we replace the words "semi-analytic" by "sub-analytic", and this enables us to overcome the difficulty above.

4) By the above, we can apply Hironaka's resolution of singularities theorem to reduce to the case where Y is a *normal crossing divisor* in X, i.e., for every $x_0 \in X$, there exists an open neighbourhood U of x_0 and a system of local coordinates $z_1, \dots, z_m$ on U such that $U \cap Y = \{z \in U\,;\, z_1 \cdots z_m = 0\}$.

2 A local study of functions in the Nilsson class

By remark 4) above, we are reduced to a local study of functions in the Nilsson class in the case where $X = D^n$ and $Y = \{z \in D^n\,;\, z_1 \cdots z_m = 0\}$. We therefore have

$$X^* = (D^*)^m \times D^{n-m} = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n\,;\, 0 < |z_i| < 1,\ 1 \leqslant i \leqslant m \\ |z_j| < 1,\ m+1 \leqslant i \leqslant n\big\}.$$

Let f be a multivalued analytic function on X^*, and let $V_{\widetilde{X}^*}$ be the vector space spanned by the branches of f, considered as functions on a covering of X^*. Let $b = (b_1, \dots, b_\mu)$ be a basis of $V_{\widetilde{X}^*}$ which serves to identify $V_{\widetilde{X}^*}$ with $\mathbb{C}^\mu$. We will repeat the argument which we used in the case of one variable. The only difference is that the fundamental group of X is now spanned by m loops $\omega_1, \dots, \omega_m$ around the origin in the first m coordinate planes, respectively. We still have $[\omega_i](b) = \mathrm{T}_i \circ b = b \circ \mathrm{M}_i$, where T_i (resp. M_i) is the monodromy transformation (resp. matrix) corresponding to $[\omega_i]$.

Since the loops $\omega_1, \dots, \omega_m$ are pairwise commutative ($\pi_1(D^{*m}) \simeq \mathbb{Z}^m$), the same is true for the matrices M_i. Therefore, there exist pairwise commuting matrices A_i such that $\mathrm{M}_i = \mathrm{e}^{2\pi i \mathrm{A}_i}$. Let us consider the matrix-valued multivalued function $z^{\mathrm{A}_i} = \mathrm{e}^{\mathrm{A}_i \log z}$. By doing an analytic continuation along ω_j ($1 \leqslant j \leqslant m$), we obtain:

$$[\omega_j]\left(z_i^{\mathrm{A}_i}\right) = \begin{cases} z_i^{\mathrm{A}_i}\mathrm{M}_i & \text{if } j = i \\ z_i^{\mathrm{A}_i} & \text{if } j \neq i. \end{cases}$$

On the other hand, it is clear that each matrix $z_i^{-\mathrm{A}_i}$ defines an automorphism of the sheaf $\mathscr{O}^\mu_{\widetilde{X}^*}$. We therefore have a homomorphism γ from the sheaf $\mathscr{O}^\mu_{\widetilde{X}^*}$ to the sheaf $\mathscr{O}_{\widetilde{X}^*}$ by setting $\gamma = \beta \circ z^{-\mathrm{A}_1} \circ \cdots \circ z_m^{-\mathrm{A}_m}$, where β is the homomorphism from the sheaf $\mathscr{O}^\mu_{\widetilde{X}^*}$ to $\mathscr{O}_{\widetilde{X}^*}$ defined by $\beta(g_1, \dots, g_\mu) = \sum_{i=1}^\mu g_i b_i$.

By applying γ to a μ-plet of *single-valued* analytic functions on X^*, and by doing analytic continuations along each ω_i, we have

$$\begin{aligned}[\omega_i]\left(\gamma\left(g_1,\ldots,g_\mu\right)\right) &= \omega_i\left(\beta\right)\omega_i\left(z_1^{-\mathrm{A}_1}\right)\cdots\omega_i\left(z_m^{-\mathrm{A}_m}\right)\left(g_1,\ldots,g_\mu\right)\\ &= \beta\mathrm{M}_i\mathrm{M}_i^{-1}z_1^{-\mathrm{A}_1}\cdots z_m^{-\mathrm{A}_m}\left(g_1,\ldots,g_\mu\right)\\ &= \beta z_1^{-\mathrm{A}_1}\cdots z_m^{-\mathrm{A}_m}\left(g_1,\ldots,g_\mu\right)\\ &= \gamma\left(g_1,\ldots,g_\mu\right).\end{aligned}$$

Therefore, γ defines a homomorphism: $\mathscr{O}_{X^*}^{\mu} \to \mathscr{O}_{X^*}$.

In particular, the image g_i of the constant section e_i, where e_i is the i-th vector in the canonical basis of $\mathbb{C}^\mu$, is a single-valued analytic function on X^*.

On the other hand, we can see by *successively* putting the matrices $\mathrm{A}_m, \mathrm{A}_{m-1}, \ldots$ into Jordan canonical form, that each component of the vector $z_1^{\mathrm{A}_1}\cdots z_m^{\mathrm{A}_m}(v)$, $v \in \mathbb{C}^\mu$, can be written in the form

$$\sum_k z_1^{\alpha_{1,i,k}}\cdots z_m^{\alpha_{m,i,k}}\, Q_{i,k}(\log z_1,\ldots,\log z_m),$$

where $\alpha_{1,i,k},\ldots,\alpha_{m,i,k}$ are eigenvalues of $\mathrm{A}_1,\ldots,\mathrm{A}_m$, respectively, and the Q are polynomials in m variables with constant coefficients.

Putting these expressions into the relation

$$V_{\widetilde{X}^*} = \beta\left(\mathbb{C}^\mu\right) = \gamma\circ z_1^{\mathrm{A}_1}\cdots z_m^{\mathrm{A}_m}\left(\mathbb{C}^\mu\right),$$

we finally obtain

$$(2.1)\quad \gamma\circ z_1^{\mathrm{A}_1}\cdots z_m^{\mathrm{A}_m}(v) = \sum_{i=1}^{\mu} g_i \sum_k z_1^{\alpha_{1,i,k}}\cdots z_m^{\alpha_{m,i,k}} Q_{i,k}\left(\log z_1,\ldots,\log z_m\right).$$

In the case where f has moderate growth, i.e., all the b_i have moderate growth, each $\gamma(v)$, $v \in \mathbb{C}$ still has moderate growth because the $z_i^{-\mathrm{A}_i}$ do. In particular, the g_i have moderate growth, and we can assume that they are *holomorphic* on X by subtracting positive integers from the exponents $\alpha_{j,i,k}$ if necessary. We therefore have:

2.2 Lemma. *A multivalued analytic function on X^* has finite determination if and only if it can be written in the form* (2.1)*, where the g_i are holomorphic functions on X^*.*

Furthermore, f has moderate growth if and only if the functions g_i in this expression can be taken to be holomorphic on X.

Remark. One can also refine lemma 2.2 a little to obtain the "uniqueness" of the expansion (2.1). In fact, by writing the terms of $Q_{i,k}$ explicitly, one can expand a multivalued analytic function of finite determination on X^* in the form

$$(2.3)\qquad f(z) = \sum_{(\alpha,p)} g_{\alpha,p}(z)\, z_1^{\alpha_1}\cdots z_m^{\alpha_m}\log^{p_1} z_1\cdots\log^{p_m} z_m,$$

where $g_{\alpha,p} \in \mathscr{O}(X^*)$, and the summation is extended to a finite subset of $\mathscr{R} \times \mathbb{Z}^{+m}$, where $\mathscr{R}$ is a fixed family of representatives of $\mathbb{C}^m/\mathbb{Z}^m$. If, furthermore, f has moderate growth, the $g_{\alpha,p}$ can be chosen to be meromorphic on X.

In this form, let us show that the expansion (2.3) is unique. In particular, we will deduce from this that, if a function with moderate growth has an expansion of the form (2.3) with $g_{\alpha,p} \in \mathscr{O}(X^*)$, then the $g_{\alpha,p}$ are indeed meromorphic functions on X.

To prove the uniqueness, we do an induction on m. It is enough to consider only the case $m = 1$ and to prove that the sum E of the $\mathbb{C}$-vector spaces $E_{\alpha,p} = \{g(z)z^\alpha \log^p z\,;\, g \in \mathscr{O}(X^*)\}$, $p \in \mathbb{Z}^+$, $\alpha \in \mathscr{R}$, where $\mathscr{R}$ is a fixed family of representatives $\mathbb{C}/\mathbb{Z}$, is in fact a *direct sum*.

We set

$$E_\alpha = \sum_{p\in\mathbb{Z}^+} E_{\alpha,p}, \quad \alpha \in \mathscr{R}.$$

Then every E_α is contained in the subspace F_λ consisting of the generalized eigenvectors for the monodromy transformation ω in E corresponding to the eigenvalue $\lambda = \mathrm{e}^{2\pi i\alpha}$, i.e., the subspace of vectors $v \in E$ which are annihilated by a power of $(\omega - \lambda \cdot \mathbb{1})$.

On the other hand,

$$E = \sum_{\alpha\in\mathscr{R}} E_\alpha \subset \sum_\lambda F_\lambda \subset E.$$

We must therefore have $\sum_\lambda F_\lambda = E$, and the inclusion $E_\alpha \subset F_\lambda$ cannot be strict. But then the sum $E = \sum_\alpha E_\alpha$ is direct. Therefore, it suffices to prove that the sum $E_\alpha = \sum_{p\in\mathbb{Z}^+} E_{\alpha,p}$ is direct. By dividing the terms E_α and $E_{\alpha,p}$ by z^α if necessary, we reduce to showing that the sum $E_0 = \sum_{p\in\mathbb{Z}^+} E_{0,p}$ is direct.

Suppose on the contrary that there exists a finite linear relation of the form:

$$\sum_{p=0}^{k} g_p(z) \log^p z = 0, \quad g_k \neq 0.$$

Then we have

$$0 = (\omega - \lambda \cdot \mathbb{1})^k \left(\sum_{p=0}^{k} g_p(z) \log^p z\right) = k!(2\pi i)^k\, g_k(z),$$

which contradicts the fact that $g_k(z) \neq 0$.

2.4 Proposition. *Let f be a multivalued analytic function of finite determination on $X^* = X - Y$, where Y is a complex hypersurface in X. Then f has moderate growth along Y if and only if, for every analytic map $\lambda : D \to X$ such that $\lambda(0) \in Y$ and $\lambda(D^*) \cap Y = \varnothing$, $f \circ \lambda$ has moderate growth near the origin (in D^*).*

Proof. First of all, the condition $\lambda(D^*)\cap Y=\varnothing$ implies that $f\circ\lambda$ is effectively of finite determination on D^*. As always, we can assume that Y is a normal crossings divisor, and we are reduced to the case $X^*=D^{*m}\times D^{n-m}$. Then the condition is obviously necessary. Let us now prove that it is sufficient.

By lemma 2.2 and the remark which follows it, we have

$$f(z)=\sum_{(\alpha,p)} g_{\alpha,p}(z)z_1^{\alpha_1}\cdots z_m^{\alpha_m}\log^{p_1} z_1\cdots\log^{p_m} z_n, \tag{2.5}$$

where (α,p) runs over a finite subset of $\mathscr{R}\times\mathbb{Z}^{+m}$, and $g_{\alpha,p}\in\mathscr{O}(X^*)$.

Now we set $z=(z_1,\dots,z_m)=(z_1,z')$, where $z'=(z_2,\dots,z_m)\in D^{*m-1}\times D^{n-m}$. For each $a'\in D^{*m-1}\times D^{n-m}$, we do indeed have an analytic map $\lambda_{a'}:D\to X$ such that $\lambda_{a'}(0)\in Y$, and $\lambda_{a'}(D^*)\cap Y=\varnothing$ by simply defining $\lambda_{a'}(t)=(t,a')$, $t\in D$.

For each fixed value ξ,j of α_1, p_1 in the sum (2.5), we set

$$\varphi_{\xi,j}(t,a')={\sum_{(\alpha,p)}}' g_{\alpha,p}(t,a')a_2^{\alpha_2}\cdots a_m^{\alpha_m}\log^{p_2} a_2\cdots\log^{p_m} a_m,$$

where the sum $\sum'$ is extended to all indices α,p such that $\alpha_1=\xi$, $p_1=j$.

Since $f\circ\lambda_{a'}$ has moderate growth by hypothesis, each $\varphi_{\xi,j}$ is a meromorphic function of t for fixed a' by the remark after lemma 2.2.

On the other hand, let $g_{\alpha,p}(t,a')=\sum_{i=-\infty}^{\infty} b^i_{\alpha,p}(a')/t^i$ be the Laurent expansion of $g_{\alpha,p}$ on D^*, where $b^i_{\alpha,p}\in\mathscr{O}(D^{*m-1}\times D^{n-m})$. Then we observe that there exists an integer k such that for all $i\leqslant k$,

$$\varphi^i_{\xi,j}(a')\equiv{\sum_{(\alpha,p)}}' b^i_{\alpha,p}(a)a_2^{\alpha_2}\cdots a_m^{\alpha_m}\log^{p_2} a_2\cdots\log^{p_m} a_m=0,\qquad \forall\, a'\in D^{*m-1}\times D^{n-m}.$$

Suppose, in effect, that the contrary is true. Then the set

$$\{z'\in D^{*m-1}\times D^{n-m}\,;\,\exists\, k \text{ such that } \varphi^i_{\xi,j}(z')=0,\ \forall\, i\leqslant k\}$$

is of measure zero, since it is a countable union of sets of measure zero. But this is absurd since for every fixed a' in $D^{*m-1}\times D^{n-m}$, $\varphi_{\zeta,j}$ is a meromorphic function of t.

Since the indices $p_2,\dots,p_m$ in the sum $\sum'$ are all distinct and the indices $\alpha_2,\dots,\alpha_m$ are not congruent mod $\mathbb{Z}^{m-1}$, we conclude, by the remark following lemma 2.2, that

$$\forall\, i\leqslant k,\ \forall\, a'\in D^{*m-1}\times D^{n-m},\quad b^i_{\alpha,p}(a')=0.$$

In other words, $z_1=0$ is a polar singularity of $g_{\alpha,p}(z_1,z')$ viewed as a function of (z_1,z').

By subtracting integers from α_1 as above if necessary, we can always assume that the $g_{\alpha,p}$ are holomorphic on the component $\{z_1=0\}$ of the divisor Y. We then repeat the argument on the component $\{z_2=0\}$, and so on in the same way. $\square$

X

Analyticity of integrals depending on parameters

1 Single-valued integrals

Let $\pi : X \to T$ be a smooth map of complex analytic manifolds, that is, such that the tangent map is surjective everywhere. By the implicit function theorem, we can choose a local coordinate system in the neighbourhood of every point of X such that π is defined by $\pi(x_1, \ldots, x_n; t_1, \ldots, t_k) = (t_1, \ldots, t_k)$.

Intuitively, $x_1, \ldots, x_n$ will be integration variables, and $t_1, \ldots, t_k$ will be parameters. The integrands will be relative differential forms of degree p, i.e., differential forms ω on X such that, locally,

$$\omega = \sum_{|I|=p} g_I(x,t)\, dx^I,$$

where $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for $I = (i_1, \ldots, i_p)$, and where g_I is an analytic function on an open subset of X on which ω is defined by the formula above.

Let $d_{X/T}$ denote the exterior differential for relative differential forms:

$$d_{X/T} : \Omega^p(X/T) \longrightarrow \Omega^{p+1}(X/T).$$

In the sequel, we shall only integrate closed forms, i.e., those satisfying $d_{X/T}\omega = 0$. Note that forms of degree n are automatically closed.

As a result, the integral of such a form over a cycle of dimension p will only depend on the homology class of the cycle. This is why we will view it as the integral of a closed relative differential form over a homology class $h(t) \in H_p(X_t)$, $X_t = \pi^{-1}(t)$.

We shall require that this class depends continuously on t, in the sense to be made precise in appendix A.

1.1 Proposition. *Let $\omega \in \Omega^p(X/T)$ satisfying $d_{X/T}\omega = 0$, and let $h(t) \in H_p(X_t)$ depend continuously on t. Then $f(t) = \int_{h(t)} \omega$ is a holomorphic function on T.*

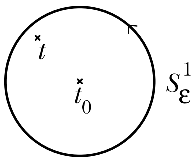
F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_10,

Proof. By Hartogs' theorem, we can reduce to the case $k = 1$. Therefore we need to prove that $\partial f/\partial\bar{t} = 0$. The difficulty in calculating the derivative $\partial f/\partial\bar{t}$ comes from the fact that not only the form being integrated, but also the integration cycle $h(t)$, depends on t. We reduce to the case of an integral over a cycle which is *independent of* t by considering the composed homomorphism ("the Leray coboundary"):

$$\delta_t : H_p(X_t) \longrightarrow H_{p+2}(X, X - X_t) \xrightarrow{\partial_t} H_{p+1}(X - X_t),$$

where ∂_t is the boundary homomorphism in homology, and the isomorphism on the left is the one explained in appendix A.

Explicitly, if $h(t)$ is the class of a cycle $\sum c_i s_i$, then $\delta_t h(t)$ is the class of a cycle $\sum c_i(S^1_\varepsilon \otimes s_i)$, where S^1_ε is a small circle of radius ε which winds once around t in T in the positive direction. Observe that if t moves in a sufficiently small neighbourhood of a point t_0, the cycle which represents the class $\delta_t h(t)$ can thus be chosen to be independent of t.



Now, "Leray's residue formula" allows us to rewrite our integral as an integral over $\delta_t h(t)$:

$$f(t) = \int_{h(t)} \omega = \frac{1}{2\pi i}\int_{\delta_t h(t)} \frac{dt' \wedge \omega}{t' - t},$$

which then enables us to differentiate under the integral, and gives $\partial f/\partial\bar{t} = 0$. □

Exercise. Prove Leray's residue formula by letting ε tend to zero in the cycle $\sum c_i(S^1_\varepsilon \otimes s_i)$ which was constructed above (the proof is similar to the proof of Cauchy's residue formula in the case of a single variable).

2 Multivalued integrals

In practice, it is often difficult to apply proposition 1.1 directly, because the integral $f(t)$ is only given in the neighbourhood of a point $t_0 \in T_0$. In other words, we are not given a section of $H_p(X/T)$, but only a local section. The problem is therefore to know how to continue this germ to give a global section on T, or perhaps on a slightly smaller subset T^*.

A situation which one often encounters in practice is the following:

Let $\pi : X \to T$ be a proper map of complex analytic manifolds, and let ω be a closed relative differential form on $X^* = X - Y$, where Y is a hypersurface on X (we can assume that $\pi|X^*$ is *smooth* but this is not very important).

The reason why we are interested in this situation is because of:

2.1 Lemma. *There exists a hypersurface $\Sigma \subset T$ such that π: $X^*_{T^*} = \pi^{-1}(T^*) \to T^* = T - \Sigma$ is a* locally trivial $\mathscr{C}^\infty$ *fibration.*

As a consequence of lemma 2.1, each $H_p(X^*_{T^*}/T^*)$ is a *locally constant* sheaf. It follows that every local section of this sheaf extends to give a *multivalued global section*. We therefore see by applying proposition 1.1 to a covering $\widetilde{T}^*$ of T^*, that $f(t)$ is multivalued and analytic on T^*.

More generally, we could equally have assumed that ω is multivalued on X^*, i.e., $\omega \in \Omega^p(\widetilde{X}^*/T)$ and $d_{\widetilde{X}^*/T}\omega = 0$, where $\widetilde{X}^*$ is a covering of X^*.

The map $\widetilde{X}^*_{T^*} \to T^*$ is thus also a locally trivial $\mathscr{C}^\infty$ fibration (since a covering of a locally trivial fibration is a locally trivial fibration). For the integral to make sense, we can take h to be a multivalued section (on T^*) of the locally constant sheaf $H_p(\widetilde{X}^*_{T^*}/T^*)$. An analogous argument to the previous one shows that under these hypotheses we have the:

2.2 Proposition. $f(t) = \int_{h(t)} \omega$ *is a* multivalued *analytic function on T^*.*

Outline of the proof of lemma 2.1. As we are only interested in X^*, we can assume as usual that Y is a normal crossing divisor, i.e., $Y = Y_1 \cup \cdots \cup Y_k$, where the Y_i are hypersurfaces without singularities, in general position. We can moreover choose a system of local coordinates such that the hypersurfaces Y_i are coordinate hypersurfaces. Then, for each $I \subset \{1, \ldots, k\}$, $Y^I = \bigcup_{i \in I} Y_i$ is a submanifold of codimension $|I|$. Observe that $Y^\varnothing = X$. Let $S_I = \{x \in Y^I; T_X(\pi|Y^I) \text{ is not surjective}\}$ and let $S = \bigcup_I S_I$. Then $\Sigma = \pi(S)$ is an analytic subset of T since π is proper, and on the other hand $\pi(S) \neq T$ by Sard's lemma. We can therefore show that Σ is the hypersurface which satisfies the conclusion of lemma 2.1.

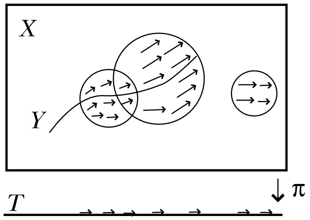
With this choice of Σ, we are in effect reduced to proving the following: let $\pi : X \to T$ be a proper map of real manifolds, and let $Y = Y_1 \cup \cdots \cup Y_k$ be a divisor of smooth hypersurfaces in X with normal crossings such that $\pi|X - Y$ is smooth, and such that for each $I \subset \{1, \ldots, k\}$, $\pi|Y^I : Y^I \to T$ has maximal rank. Then there is a diffeomorphism $\varphi : X \to X_0 \times T$, where X_0 is a real analytic manifold, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "\pi"'] & & X_0 \times T \arrow[dl, "\text{canonical projection}"] \\ & T & \end{tikzcd}$$

and furthermore, $\varphi(Y) = Y_0 \times T$, where $Y_0 \subset X_0$ (or, in other words, the pair (X, Y) is locally trivial on T).

We will only give a proof of this fact in the case where $\dim T = 1$.

Therefore, let $\{U_i\}$ be a locally finite covering of X with local charts. By hypothesis, we can construct a vector field v_i on each U_i such that the vector v_i at each point of Y^I is a tangent vector of Y^I, and such that its projection onto T under $d\pi$ is the unit vector field on T. Let $\{\varphi_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$, and let $v = \sum \varphi_i v_i$. The sought-after diffeomorphism φ is then given by integrating this vector field.



2.3 Proposition. *If, in proposition 2.2, ω is of finite determination, then $f(t)$ is also of finite determination.*

Proof. Let $\mathscr{V}$ be the locally constant sheaf of finite-dimensional vector spaces on X^* spanned by the branches of ω. Let $\mathscr{V}_{X_t^*}$ be the restriction of $\mathscr{V}$ to X_t^*, and let $\mathscr{V}_{\widetilde{X}_t^*}$ be the inverse image of $\mathscr{V}$ under the (not necessarily connected) covering $\widetilde{X}_t^* \to X_t^*$ (which is induced by the covering $\widetilde{X}^* \to X^*$).

Then $\mathscr{V}_{X_t^*}$ and $\mathscr{V}_{\widetilde{X}_t^*}$ are locally constant sheaves, and we have the following commutative diagram:

$$\begin{array}{ccccc} \mathscr{V}_{\widetilde{X}_t^*} & \dashrightarrow & \mathscr{V}_{X_t^*} & \dashrightarrow & \mathscr{V} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X}_t^* & \longrightarrow & X_t^* & & X^* \end{array}$$

Recall that for all $x \in X^*$, the fibre $\mathscr{V}_x$ of $\mathscr{V}$ over the point x is nothing other than the vector space over $\mathbb{C}$ spanned by $\omega_{\widetilde{x}}$, the value of the form ω at the point $\widetilde{x}$, where $\widetilde{x}$ runs over the set of all points of $\widetilde{X}^*$ above x. Thus for every $\widetilde{x} \in \widetilde{X}_t^*$, $\widetilde{\omega}(\widetilde{x}) = \omega_{\widetilde{x}}$ is indeed an element in the fibre of $\mathscr{V}_{\widetilde{X}_t^*}$ over $\widetilde{x}$. In other words, $\widetilde{x} \to \widetilde{\omega}(\widetilde{x})$ is a privileged section of $\mathscr{V}_{\widetilde{X}_t^*}$ determined by ω. Using this privileged section, we obtain a homomorphism from the constant sheaf to the sheaf $\mathscr{V}_{\widetilde{X}_t^*}$, simply defined by $c \to c\widetilde{\omega}$, which in turn induces a homomorphism of the homology sheaves $H_p(\widetilde{X}_t^*, \mathbb{C}) \to H_p(\widetilde{X}_t^*, \mathscr{V}_{\widetilde{X}_t^*})$ on T^*.

On the other hand, the covering $\widetilde{X}_t^* \to X_t^*$ also induces a homomorphism of sheaves $H_p(\widetilde{X}_t^*, \mathscr{V}_{\widetilde{X}_t^*}) \to H_p(X_t^*, \mathscr{V}_{X_t^*})$ (see appendix B).

The composition of these two homomorphism gives a homomorphism of sheaves $H_p(\widetilde{X}_t^*, \mathbb{C}) \to H_p(X_t^*, \mathscr{V}_{X_t^*})$ and proposition 2.3 is simply a consequence of proposition 2.4 below. $\square$

2.4 Proposition. *Let $\pi : X \to T$, and let X^*, T^* be as in proposition 2.3. Let $\mathscr{V}$ be a locally constant subsheaf of $\Omega^p(X^*/T^*)$ of finite-dimensional vector spaces. Let $h_{\mathscr{V}}$ be a* multivalued section *on T^* of the homology sheaf of X^* over T^* in dimension p, with coefficients in $\mathscr{V}$. Then $f(t) = \int_{h_{\mathscr{V}}(t)}$ is a multivalued analytic function of* finite determination *on T^*.*

Proof. For each $t \in T^*$, suppose that $h_{\mathscr{V}}(t)$ is defined by the cycle $\sum a_i s_i$, where $s_i : \Delta_p \to X_t^*$ is a singular simplex of X_t^*, and a_i is a section of the sheaf $s_i^{-1}(\mathscr{V})$ on Δ_p (the inverse sheaf of the sheaf $\mathscr{V}$ under the map s_i). Then, by definition

$$f(t) = \int_{h_{\mathscr{V}}(t)} = \sum_i \int_{s_i} a_i .$$

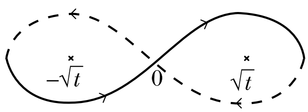
The proof of proposition 2.4 is now immediate because the compact manifold X_t^* (π is proper) has a finite triangulation, and the inverse sheaves (or rather the vector spaces) $s_i^{-1}(\mathscr{V})$ are finite-dimensional. □

Now we are ready to state Nilsson's theorem, whose proof will be sketched in the following chapter.

Nilsson's theorem. *If, in addition to the hypotheses of proposition 2.3, ω has* moderate growth along Y*, i.e., the local coefficients of ω are of moderate growth, then $f(t)$ has* moderate growth along Σ.

3 An example

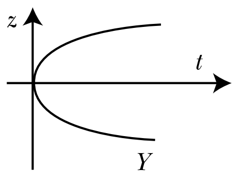
Consider the integral $f_\alpha(t) = \int_{h(t)} (z^2 - t)^\alpha dz$, $\alpha \in \mathbb{C}$, where $h(t)$ is the homology class of the cycle defined as follows:



($h(t)$ is represented by the oriented solid and dashed lines).

Here we have $T = \mathbb{C}$, $X = \mathbb{C} \times T$, and π is the canonical projection from X onto T. Strictly speaking, X should really be the product of the Riemann sphere with T for π to be proper, but we gain nothing by considering the point at infinity, and we omit it here.

$$Y = \{(z,t) \in X;\ z^2 - t = 0\}$$



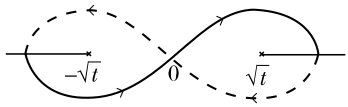
already has no singularities, and the only point of Y where $\pi|Y$ is not a submersion is $(0,0)$. We thus have $\Sigma = \pi\{(0,0)\} = \{0\}$. Therefore $T^* = \mathbb{C} - \{0\}$, and

$$\begin{aligned} X^* &= \left\{(z,t) \in \mathbb{C}^2;\ z^2 - t \neq 0\right\} \\ \omega &= (z^2 - t)^\alpha dz. \end{aligned}$$

Proposition 2.2 says that f_α is a multivalued analytic function on T, which immediately follows from the explicit formula

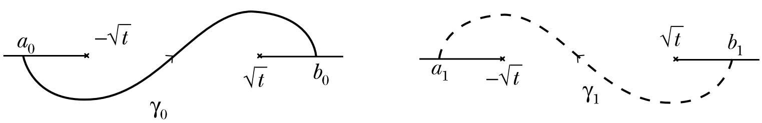
$$f_\alpha(t) = \frac{-2i\pi^{3/2}\, t^{\alpha+1/2}}{\Gamma(-\alpha)\,\Gamma\,(\alpha+3/2)}.$$

Nevertheless, if we do not wish to calculate this integral directly, then studying its monodromy already gives its properties. Therefore consider the complex plane with the two cuts indicated by the diagram:



When t winds once around the origin in the positive direction, $\sqrt{t}$ becomes $-\sqrt{t}$, and so $h(t)$ becomes $-h(t)$. On the other hand, by considering the change in the value of each branch of $(z^2-t)^\alpha$ at $z=0$, we see that it is multiplied by $\mathrm{e}^{2\pi i\alpha}$. Thus every branch of $f(t)$ is multiplied by $-\mathrm{e}^{2\pi i\alpha} = \mathrm{e}^{2\pi i(\alpha+1/2)}$. This proves that f is a multivalued function, even in the case where α is a negative integer (in this case, $(z^2-t)^\alpha$ is single-valued with polar singularities), and explains the presence of the power $t^{\alpha+1/2}$ in the explicit formula.

In order to illustrate proposition 2.3, we must consider homology groups with values in local systems of coefficients (i.e., taking values in a locally constant sheaf of vector spaces). Let X_t^* be the Riemann surface of the function $(z^2-t)^\alpha$. Since the sections of $\mathscr{V}_{\widetilde{X}_t^*}$ in the cut plane form a vector space of dimension 1, we can identify them with the complex numbers. Next, let us consider the chain $1\cdot\gamma_0 - \mathrm{e}^{-2\pi i\alpha}\gamma_1$ in $C_1(X_t^*, \mathscr{V}_{X_t{}^*})$, where γ_0 and γ_1 are the singular simplices in X_t^* indicated in the following diagram:



This chain is in fact a cycle, since

$$\begin{aligned} \partial\left(1\cdot\gamma_0 - \mathrm{e}^{-2\pi i\alpha}\,\gamma_1\right) &= b_0 - a_0 - \mathrm{e}^{-2\pi i\alpha}\,(b_1 - a_1) \\ &= \left(b_0 - \mathrm{e}^{-2\pi i\alpha}\,b_1\right) - \left(a_0 - \mathrm{e}^{-2\pi i\alpha}\,a_1\right). \end{aligned}$$

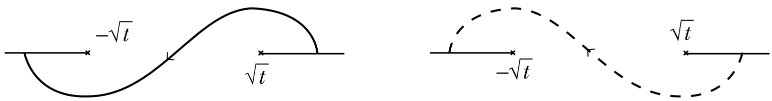
As a result, this cycle is nothing other than the image of the cycle h considered earlier under the composed homomorphism:

$$C_1\big(\widetilde{X}_t^*, \mathbb{Z}\big) \longrightarrow C_1\big(\widetilde{X}_t^*, \mathbb{C}\big) \longrightarrow C_1\big(\widetilde{X}_t^*, \mathscr{V}_{\widetilde{X}_t^*}\big) \longrightarrow C_1\big(X_t^*, \mathscr{V}_{X_t^*}\big).$$

We can also say that the homology class of this cycle in $H_1(X_t^*, \mathscr{V}_{X_t^*})$ is represented by the cycle $h(t)$. This is why we will still denote this homology class by $h(t)$. We therefore have, by definition:

$$f_\alpha(t) = \int_{h(t)} \big(z^2 - t\big)^\alpha \; dz = \int_{h(t)} .$$

Now consider the monodromy defined by a loop ω which winds once around the origin in the positive direction



We have

$$\begin{aligned} [\omega](\gamma_0(t)) &= -\mathrm{e}^{2\pi i\alpha}\, \gamma_0(t) \\ [\omega](\gamma_1(t)) &= -\mathrm{e}^{2\pi i\alpha}\, \gamma_1(t), \end{aligned}$$

which gives

$$[\omega]\,\big(\gamma_0(t) - \mathrm{e}^{-2\pi i\alpha}\, \gamma_1(t)\big) = -\mathrm{e}^{2\pi i\alpha}\, \gamma_0(t) + \gamma_1(t),$$

and thus

$$[\omega](h(t)) = \mathrm{e}^{2\pi i(\alpha+1/2)}\, h(t),$$

and therefore

$$[\omega]\big(f_\alpha(t)\big) = \mathrm{e}^{2\pi i(\alpha+1/2)}\, f_\alpha(t).$$

We thus recover the change in $f_\alpha(t)$ under the action of $[\omega]$.

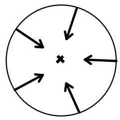
XI

Sketch of a proof of Nilsson's theorem

Owing to the results of the preceding chapter, we are reduced to proving the theorem when $\dim T = 1$, $T = D$, $T^* = D^*$. We can always assume that $Y \supset \pi^{-1}(0)$ (by adding a component to Y if necessary), and that Y is a *normal crossing divisor* (cf. the preceding chapter).

With these conditions, it is easy to construct a real $\mathscr{C}^\infty$ vector field ξ on X which is tangent to Y, and which is compatible with the projection π and projects onto the vector field

$$\xi = -\left(t\,\frac{\partial}{\partial t} + \bar{t}\,\frac{\partial}{\partial \bar{t}}\right)$$



To see this, take a local coordinate system $(z_1, \ldots, z_n)$ on X, where π is given by $t = z_1^{a_1} \cdots z_m^{a_m}$ $(m \leqslant n)$, and Y by:

$$Y = \{z; z_1 \cdots z_p = 0\} \quad (m \leqslant p \leqslant n).$$

Then we can take ξ to be the real analytic vector field

$$\xi = -\frac{1}{a_1}\left(z_1\frac{\partial}{\partial z_1} + \bar{z}_1\,\frac{\partial}{\partial \bar{z}_1}\right).$$

We can then glue the various local vector fields obtained in this way using a $\mathscr{C}^\infty$ partition of unity.

Note. Here, as in several other places in the proof, vector fields are identified with derivations in the ring of functions on a manifold.

By integrating this vector field, we obtain, over the integral curves of the vector field ξ

$$t(\tau) = t_0 \mathrm{e}^{-\tau},$$

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a family of diffeomorphisms which are parametrized by τ

$$\varphi_\tau : X_{t_0} \longrightarrow X_{t(\tau)}.$$

Our integral can thus be written

$$f(t(\tau)) = \int_{h(t(\tau))} \omega = \int_{h(t_0)} \varphi_\tau^* \omega,$$

and the problem of showing that f has moderate growth reduces to finding an upper bound for $f(t(\tau))$ of the form

$$|f(t(\tau))| \leqslant C\,\mathrm{e}^{k\tau},$$

which is locally single-valued with respect to t_0 as t_0 travels around the unit circle.

By specifying, for example, a Riemannian metric on X, one can speak of the "norm" $\|\ \|_z$ of a differential form at a point $z \in X$. Our problem reduces to bounding $\|\varphi_\tau^*\omega\|_z$ above, exponentially in τ as $\tau \to \infty$. The upper bound must be single-valued in z as t travels along the support of a cycle in the class $h(t_0)$, and single-valued in t_0 as t_0 travels along a finite arc of the unit circle.

Now, since the vector field ξ is Lipschitz (because it is $\mathscr{C}^\infty$ and has compact support), a classical inequality from the theory of differential equations allows us to state that the diffeomorphism φ_τ dilates distances by at most an exponential factor $\mathrm{e}^{\lambda\tau}$ (where λ is the Lipschitz constant of the vector field). The norm of the linear tangent map of φ_τ at every point is therefore less than $\mathrm{e}^{\lambda\tau}$, which enables us to write the norm of a differential form ω of degree p as:

$$\|\varphi_\tau^*\omega\|_z \leqslant \mathrm{e}^{p\lambda\tau}\|\omega\|_{\varphi_\tau}(z).$$

The factor $\mathrm{e}^{p\lambda\tau}$ is indeed of the required type, and *it only remains to bound* $\|\omega\|_{\varphi_\tau}(z)$ *from above.*

Lemma 1. *In the neighbourhood of a point where Y is given by a local equation s, we have*

(i) $$|\xi \cdot |s|| < \lambda |s|$$

(ii) $$|\xi \cdot \operatorname{Arg} s| < \lambda$$

where $\xi \cdot f$ is the derivative of f along ξ.

This lemma is an immediate consequence of the fact that ξ is a Lipschitz vector field which is tangent to Y. Intuitively, it means that the integral curves of the vector field

(i) approach Y with at most exponential speed,
(ii) wind around Y with bounded angular speed.

Observe that nothing prevents the integral curves from spiralling around Y, which means that the moderate growth hypothesis for ω cannot be used without doing some extra work. This extra work will consist of the following: we construct a *finite semi-analytic triangulation* (K, L) *of the pair* (X, Y), such that the star of each vertex of L in K is included in the domain of a local chart on which the bounds of lemma 1 are satisfied. Let $\widetilde{K-L}$ denote the set of lifts of simplices of $K-L$ in $\widetilde{X}^* = \widetilde{X-Y}$. For every pair of simplices k, k' of $\widetilde{K-L}$, let $\Delta_{kk'}$ denote the *gap between the two simplices* k and k', defined to be the minimal length of a chain of simplices joining k and k', i.e., the length of a sequence of simplices of $\widetilde{K-L}$ such that two consecutive simplices are adjacent. Let us fix a simplex k_0 in $\widetilde{K-L}$.

Lemma 2. *There exist $c > 0, a > 1, p \in \mathbb{N}$ such that for all $k \in \widetilde{K-L}$, $\|\omega\|_z$ is bounded in k by*

$$\|\omega\|_z \leqslant c \frac{a^{\Delta_{kk_0}}}{\{\mathrm{dist}(z, Y)\}^p}$$

*where "*dist*" denotes the distance with respect to the chosen Riemannian metric.*

Proof. In each simplex $\underline{k} \in K-L$, we choose a basis $b_{\underline{k}}$ for the vector space of determinations of ω on $\underline{k}$. For every pair of contiguous simplices $(\underline{k}, \underline{k}')$, there is a change of basis matrix $A_{\underline{k},\underline{k}'}$. The lemma is proved by observing that every "basis branch" of ω satisfies an upper bound on $\underline{k}$:

$$\|\omega\|_z < \frac{C_{\underline{k}}}{\mathrm{dist}(z, Y)^{p_{\underline{k}}}} \quad \text{(the moderate growth hypothesis for } \omega\text{)},$$

and by setting

$$C = \sup_{\underline{k} \in K-L} C_{\underline{k}}, \quad p = \sup_{\underline{k} \in K-L} p_{\underline{k}}, \quad a = \sup_{\underline{k}, \underline{k}' \in K-L} \|A_{\underline{k},\underline{k}'}\|$$

(recall that $K-L$ is a finite triangulation). $\square$

By lemma 2, and taking into account lemma 1(i) (which ensures that $\mathrm{dist}(z(\tau), Y)$ decreases at most exponentially in τ), we will obtain an upper bound of the desired type if we show that $\Delta_{k(\tau),k_0}$ has *at most linear* growth in τ (where we denote the simplex which contains the point $z(\tau) = \varphi_\tau(z_0)$ by $k(\tau)$).

Lemma 3. *There exist positive constants C and C' such that along every integral curve $z(\tau)$ of the vector field ξ, the function $\Delta_{k(\tau),k_0}$ satisfies the upper bound:*

$$\Delta_{k(\tau),k_0} \leqslant C\tau + C'.$$

Proof. Let $j : \mathbb{R} \to X$ be an integral curve of the vector field ξ. Let $\mathscr{K}_0$ (resp. $\mathscr{L}_0$) be the set of *vertices* of the triangulation K (resp. L). For every $s \in \mathscr{K}_0$, consider the open set $j^{-1}(\mathrm{st}(s))$ in $\mathbb{R}^+$, where $\mathrm{st}(s)$ denotes the open subset of X defined by the *star of s*. This gives a covering of $\mathbb{R}^+$ by *open intervals* (to each $s \in \mathscr{K}_0$ there corresponds a certain number of disjoint open intervals whose union is $j^{-1}(\mathrm{st}(s))$). From this, we extract a covering consisting of *maximal* open intervals. Then it is not difficult to see that the covering obtained in this way is *locally finite*, and even better: the lengths of the intervals which make up the covering are bounded below by a fixed number δ/v, where $v = \sup|\xi|$ (= maximal speed of the trajectory) and $\delta =$ the "minimal diameter" of a star.

It remains to check that on each of these intervals, the variation $\Delta_{kk'}$ is bounded above by a fixed integer which is independent of the interval:

1) This is obvious for $s \in \mathscr{K}_0 - \mathscr{L}_0$, since the star of s is then a simply-connected open subset of $K-L$, and each connected component of $K - L|\mathrm{st}(s)$ is isomorphic to $\mathrm{st}(s)$, so that the variation $\Delta_{kk'}$ is bounded above by the number of simplices in the star;

2) For $s \in \mathscr{L}_0$, the upper bound follows from lemma 1(ii) and from the following easy lemma:

Lemma 4. *There exist constants C_1 and $C_2 > 0$ such that the following upper bound is valid for every pair of simplices k, k' of $\widetilde{K-L}$ whose projections $\underline{k}, \underline{k}'$ in $K - L$ both contain the same vertex s of L in their closures:*

$$\Delta_{kk'} \leqslant C_1 \left(|\mathrm{Arg}\, z_1|_{k,k'} + \cdots + |\mathrm{Arg}\, z_m|_{k,k'} \right) + C_2,$$

where $z_1 \cdots z_m$ is a local equation for Y (a normal crossing divisor) in the star of s, and $|\,\mathrm{Arg}\, z_1|_{k,k'}$, denotes the supremum of the variation in the argument of the complex number z_i along a path joining k and k' whose projection remains in the star of s. □

XII

Examples: how to analyze integrals with singular integrands

Suppose that we are given a function $f(t)$ defined by an integral. If the integral satisfies, for example, the conditions of Nilsson's theorem, then we know that the function $f(t)$ is in the Nilsson class. This means that if, for example, t is a single variable, then in the neighbourhood of each singular point, $f(t)$ has an expansion as a series of complex powers and logarithms of the type studied in chapter VIII.

In concrete problems, we will often want to have more precise information about this expansion, and to know its singular part explicitly, for example.

Some examples will help to suggest some general ideas on the manner in which one can obtain such information.

1 First example

Consider the integral

$$f_\alpha(t) = \int_{\mathbb{R}^n} u(x,t)\left(x^2 - t\right)^\alpha dx, \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$, $x^2 = \sum_{i=1}^n x_i^2$, $dx = dx_1 \cdots dx_n$, $\alpha \in \mathbb{C}$, and $u(x,t) \in \mathscr{A}(\mathbb{R}^{n+1})$ (the set of analytic functions on $\mathbb{R}^{n+1}$). We suppose, furthermore, that u decreases sufficiently fast at infinity for the integral (1.1) to converge absolutely and uniformly with respect to t. Then, for $t < 0$, we have a function which is indeed analytic in t.

Now the problem is to define an *analytic continuation* of $f_\alpha(t)$ for complex t, and to study the *singularity* of f at the point $t = 0$.

To do this, we can try to replace $\mathbb{R}^n$ by a "cycle" of dimension n which does not meet the paraboloid defined by the singularity of the integrand:

$$z^2 - t = 0.$$

In order to draw pictures in the case of several complex variables ($n > 1$), it is convenient to represent each point $z = x + iy \in \mathbb{C}^n$ by a pair given by the

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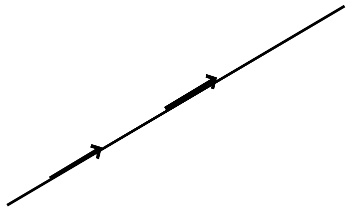
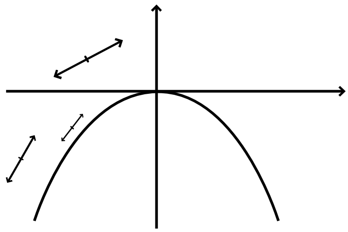
point $x \in \mathbb{R}^n$ and the vector y based at x. In other words, $\mathbb{C}^n$ is represented by the tangent bundle of $\mathbb{R}^n$ and each subset of $\mathbb{C}^n$ is represented by a subset of this bundle.

Examples.

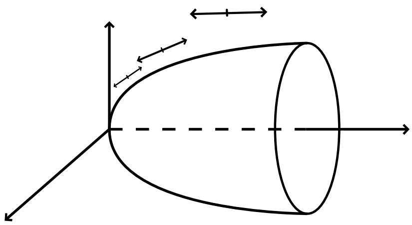
1. The complex line $az_1 + bz_2 + c = 0$ in $\mathbb{C}^2$ with real coefficients a, b, c can be represented by the tangent bundle of the real line $ax_1 + bx_2 + c = 0$.



2. The complex parabola $z_1^2 + z_2 = 0$ in $\mathbb{C}^2$ can be represented by two opposite vector fields defined in the region of the real plane which lies outside the parabola $x_1^2 + x_2 = 0$, such that the vectors become nearly parallel and of smaller and smaller length as one approaches the parabola.



3. In the case which interests us, we can represent the singular paraboloid $z^2 - t = 0$ by two opposite vector fields which are nearly parallel to the real paraboloid $x^2 - t = 0$, and having smaller and smaller length in a neighbourhood of this paraboloid.



Intuitively, to obtain the new integration "cycle" by "displacing" $\mathbb{R}^{n+1}$ in $\mathbb{C}^{n+1}$ in such a way as to avoid the complex paraboloid $z^2 - t = 0$, it therefore suffices to take a vector field which is *transverse to the real paraboloid.*

More precisely, we have the

1.2 Definitions.

i) A *complex displacement* of $\mathbb{R}^n$ in $\mathbb{C}^n$ is a continuous map $\sigma : \mathbb{R}^n \to \mathbb{C}^n$ such that $\mathrm{pr} \circ \sigma = \mathbb{1}_{\mathbb{R}^n}$, where pr is the natural projection from $\mathbb{C}^n$ to $\mathbb{R}^n$. In other words, a displacement is a map of the form $x \longmapsto x + i\underline{\sigma}(x)$, where $\underline{\sigma}$ is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.
ii) An *infinitesimal displacement* of $\mathbb{R}^n$ is a continuous vector field on $\mathbb{R}^n$, i.e., a map of the form:

$$\begin{aligned} \theta : \mathbb{R}^n &\longrightarrow T\mathbb{R}^n = \mathbb{R}^n \oplus \mathbb{R}^n \\ x &\longmapsto (x, \underline{\theta}(x)) \,. \end{aligned}$$

iii) A *one-parameter family of displacements* is a continuous map

$$\begin{aligned} \widetilde{\sigma} : \mathbb{R}^n \times [0, \varepsilon_0) &\longrightarrow \mathbb{C}^n \\ (x, \varepsilon) &\longmapsto \sigma_\varepsilon(x) = x + i\underline{\sigma}\,(x, \varepsilon) \end{aligned}$$

such that, for all $x \in \mathbb{R}^n$, we have $\underline{\sigma}(x, 0) = 0$.
iv) Suppose, furthermore, that the map $\underline{\sigma}$ in iii) is differentiable with respect to ε. Then the *infinitesimal displacement associated to the family* $\widetilde{\sigma}$ is the displacement $x \longmapsto (x, \underline{\theta}(x))$ where, for all $x \in \mathbb{R}^n$, we set $\underline{\theta}(x) = \partial\underline{\sigma}/\partial\varepsilon(x, \varepsilon)|_{\varepsilon=0}$.

Example. Let $\theta : x \longmapsto (x, \underline{\theta}(x))$ be an infinitesimal displacement of $\mathbb{R}^n$. Then the family of displacements which are "linear in ε" given by $\mathbb{R}^n : (x, t) \mapsto x + i\varepsilon\underline{\theta}(x)$, has θ as its associated infinitesimal displacement.

Note that, in $\mathbb{R}^n$, the following notions can be identified with each other: 1) complex displacements, 2) infinitesimal displacements, 3) "linear" families of displacements. Note, however, that this identification is not intrinsic (i.e., it depends on changes of coordinates).

Given the definitions above, our problem will be solved by:

1.3 Proposition. *Let $Y_\mathbb{R}$ be a smooth hypersurface in $\mathbb{R}^n$, and let Y be its complexification. Let (σ_ε) be a family of complex displacements of $\mathbb{R}^n$ such that the corresponding infinitesimal displacement is* transverse *to $Y_\mathbb{R}$. Then for every compact set $K \subset \mathbb{R}^n$, we can find a number $\varepsilon_K > 0$ such that for all $\varepsilon > \varepsilon_K$, $\sigma_\varepsilon|_K$ avoids Y completely.*

Proof. We will limit ourselves to giving only a general idea of the proof of this proposition.

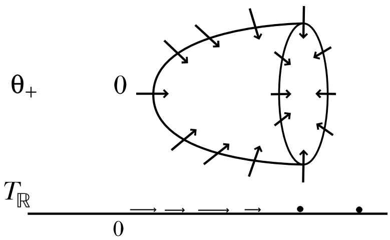
Note first of all that the notion of infinitesimal displacement corresponding to a family of complex displacements is *intrinsic*, which allows one to do a local change of coordinates to reduce to the case where Y is a hyperplane. Suppose, for the sake of argument, that for a suitably chosen open set $U \subset \mathbb{C}^n$, $Y \cap U$ coincides with the hyperplane $\{(z_1, \dots, z_n) \in U\,;\, z_1 = 0\}$. Thus, to say that the corresponding infinitesimal displacement θ is transverse to $Y_{\mathbb{R}}$ amounts to saying that the first component $\underline{\theta}_1(x)$ of $\underline{\theta}(x)$ is non-zero for $x \in Y_{\mathbb{R}}$. In other words, $d\{\underline{\sigma}_\varepsilon(x)\}_1/d\varepsilon|_{\varepsilon=0} \neq 0$.

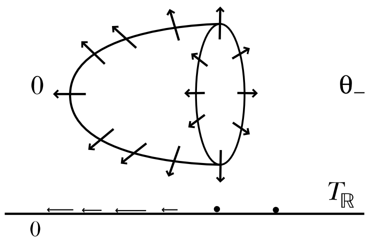
Let K be an arbitrary compact subset of $\mathbb{R}^n$ such that $K \subset U$. Then we can find a number $\varepsilon_K > 0$ such that, for all $x \in K$ and all $\varepsilon < \varepsilon_K$, we have $x + i\underline{\sigma}_\varepsilon(x) \in U$ and $d\{\underline{\sigma}_\varepsilon(x)\}_1/d\varepsilon \neq 0$. Then for every $x \in K$, $\{\underline{\sigma}_\varepsilon(x)\}_1$ is a strictly monotonic function of $\varepsilon \in [0, \varepsilon_K)$. Hence $|\{\underline{\sigma}_\varepsilon(x)\}_1| > |\{\underline{\sigma}_0(x)\}_1| = 0$ for $0 < \varepsilon < \varepsilon_K$.

Then, it is clear that $\sigma_\varepsilon|_K$ completely avoids Y. Finally, since each compact subset of $\mathbb{R}^n$ can be written as a finite union of compact sets each of which is contained in such an open set U, the proposition is proved. □

Let us apply this proposition to our problem. We are led to consider the following construction:

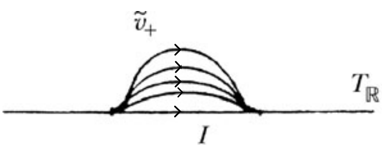
1) We construct two infinitesimal displacements θ_+ and θ_- of $\mathbb{R}^{n+1}$ in $\mathbb{C}^{n+1}$, which are transverse to $Y_{\mathbb{R}}$ and compatible with the projection onto $T_{\mathbb{R}} = \mathbb{R}$.

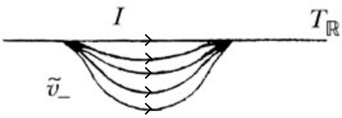




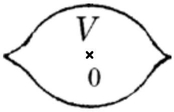
Observe that the condition of being transversal to $Y_{\mathbb{R}}$ at 0 implies that the projection of θ_+ (resp. θ_-) onto $T_{\mathbb{R}}$ is a non-zero vector field at 0 with > 0 component (resp. < 0). On the other hand, this vector field could be chosen to be zero on $T_{\mathbb{R}} - I$, where I is an open interval containing 0.

2) The two infinitesimal displacements θ_+ and θ_- can be considered as being associated to two families of displacements $\widetilde{\sigma}_+$ and $\widetilde{\sigma}_-$ which respectively project onto two families $\widetilde{v}_+$, $\widetilde{v}_-$ of displacements of $T_{\mathbb{R}}$.





We can assume that these two families $\widetilde{v}_+$ and $\widetilde{v}_-$ sweep out a complex neighbourhood V of I (as in the figure), in which each point t with imaginary part > 0 (resp. < 0) belongs to one and only one displacement of the family $\widetilde{v}_+$ (resp. $\widetilde{v}_-$).



We can also assume that the two families of displacements $\widetilde{\sigma}_+$ and $\widetilde{\sigma}_-$ are completely contained in the complex neighbourhood U of $\mathbb{R}^{n+1}$ where the function $u(z,t)$ is holomorphic. This enables us to associate to these two families of displacements two functions:

$$f_\alpha^\pm(t) = \int_{h^\pm(t)} u(z,t)\left(z^2 - t\right)^\alpha dz_1 \wedge \cdots \wedge dz_n,$$

which are analytic on the two connected components $(\operatorname{Im} t > 0)$ and $(\operatorname{Im} t < 0)$ of $V - I$, where $h^+(t)$ (resp. $h^-(t)$) denotes the homology class, in $U - Y_t$, of the cycle which is the image of the displacement of $\mathbb{R}^n$ defined by $\widetilde{\sigma}_+$ (when $\operatorname{Im} t > 0$) (resp. $\widetilde{\sigma}_-$ (when $\operatorname{Im} t < 0$)). Of course, these two analytic functions $f_\alpha^\pm(t)$ take boundary values which coincide on the negative real part of V, since for t real < 0 we obviously have $h^+(t) = h^-(t) =$ the homology class of the undisplaced $\mathbb{R}^n$.

The functions $f_\alpha^+(t)$ and $f_\alpha^-(t)$ therefore define the required analytic continuation of the integral (1) along $V - (\mathbb{R}^+ \cap V)$. The study of the discontinuity around the cut $\mathbb{R}^+ \cap V$ will enable us to continue this integral as a *multivalued* function on $V - \{0\}$, and to make the monodromy of this function explicit.

Study of the monodromy

In the preceding discussion, one must not forget that $h^+(t)$ and $h^-(t)$ should in fact be considered as homology classes with *values in the local system of coefficients* spanned by the branches of $(z^2 - t)^\alpha$. For $h^+(t)$ and $h^-(t)$ to coincide for t real and < 0, it is obviously necessary that the branch chosen for h^+ and h^- be the same on the whole half space $\{(x,t) \in \mathbb{R}^{n+1}\,;\, t < 0\}$, and thus in fact on all the exterior of the paraboloid (the part with $x^2 - t > 0$). For t real and > 0, the difference $h^-(t) - h^+(t)$ will therefore be a homology class $e(t)$ with support in a compact neighbourhood of the ball $\{x \in \mathbb{R}^n;\, x^2 - t \leqslant 0\}$. We can make this homology class $e(t)$ explicit in the following way:

The cycles $h^{\pm}(t)$ viewed "in polar coordinates"

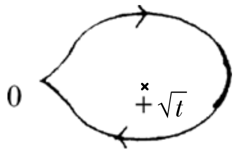
$$\begin{aligned} \mathbb{R}^+ \times S^{n-1} &\longrightarrow \mathbb{C}^n \\ (\rho,\zeta) &\longmapsto \gamma_+^{\pm}(\rho)\cdot\zeta \end{aligned}$$

where S^{n-1} is the real oriented sphere, viewed as being embedded in $\mathbb{C}^n$, and $\gamma_+^{\pm}$ are the displacements of $\mathbb{R}^+$ in $\mathbb{C}$ around the point $+\sqrt{t}$:

The cycle $e(t) = h^-(t) - h^+(t)$ viewed in polar coordinates

$$\begin{aligned} [0,1] \times S^{n-1} &\longrightarrow \mathbb{C}^n \\ (\rho,\zeta) &\longmapsto \varepsilon_t(\rho)\cdot\zeta \end{aligned}$$

where ε_t is the curve represented by the diagram:



Warning: this curve should be considered as a chain with coefficients in the sheaf of determinations of $(z^2-t)^\alpha$; one must therefore beware of the fact that the branch at the extremity of this curve differs from the branch at the origin by a factor of $\mathrm{e}^{2i\pi\alpha}$.

The monodromy can easily be deduced from this representation. In fact, as t travels along a circle around 0 in the positive direction (the loop ω), $+\sqrt{t}$ and $-\sqrt{t}$ are interchanged after travelling along two opposite semicircles, and this forces the curve γ_t^+ to be pushed upwards until it becomes γ_t^-: we thus have $[\omega]h^+(t) = h^-(t)$ (which was, in fact, *a priori* obvious). On the other hand, ε_t turns into the curve which is symmetric with respect to the origin, on the branch of $(z^2-t)^\alpha$ multiplied by $\mathrm{e}^{2\pi i\alpha}$. Because of the fact that in polar coordinates, a reflection in the origin amounts to doing an antipodal symmetry of the sphere S^{n-1}, and because this multiplies its orientation by the factor $(-1)^n$, we see that

$$[\omega]e(t) = (-1)^n\mathrm{e}^{2\pi i\alpha} \quad e(t) = \mathrm{e}^{2\pi i(\alpha+n/2)}e(t).$$

Setting

$$g_\alpha(t) = \int_{e(t)} u(z,t)\left(z^2 - t\right)^\alpha dz_1 \wedge \cdots \wedge dz_n,$$

we see that in this way, we can define a locally constant sheaf of vector spaces of dimension 2 on the open set $V - \{0\}$, spanned in a half-neighbourhood $\operatorname{Im} t > 0$ of the cut $V \cap \mathbb{R}^+$ by the two functions $f_\alpha^+(t)$ (which we simply write $f_\alpha(t)$) and $g_\alpha(t)$, with the monodromy matrix.

$$\begin{bmatrix} 1 & 1 \\ 0 & \mathrm{e}^{2\pi i(\alpha+n/2)} \end{bmatrix}.$$

We immediately deduce that $g_\alpha(t) = v_\alpha(t)t^{\alpha+n/2}$, $v_\alpha \in \mathscr{O}(D^*)$.

On the other hand, since $(\mathrm{e}^{2\pi i(\alpha+n/2)} - 1)f_\alpha - g_\alpha$ is invariant under monodromy, it also belongs to $\mathscr{O}(D^*)$. We distinguish two cases:

1) $\alpha + n/2 \notin \mathbb{Z}$: $f_\alpha = \dfrac{g_\alpha}{\mathrm{e}^{2\pi i(\alpha+n/2)} - 1} \mod \mathscr{O}(D^*)$;

2) $\alpha + n/2 \in \mathbb{Z}$: the monodromy matrix is therefore

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We deduce that $f_\alpha - g_\alpha \log t/2\pi i$ is invariant under monodromy, i.e., $f_\alpha = g_\alpha \log t/2\pi i \mod \mathscr{O}(D^*)$.

In fact one can find an upper bound for the integral (1.1) to obtain more precise information on the asymptotic nature of f and g near the origin.

The behaviour of g_α near the origin

Let us suppose that t is sufficiently small that we can choose the curve ε_t to be the circle centred at $+\sqrt{t}$ with radius $\sqrt{t}$. Then $g_\alpha(t)$ is given by:

$$g_\alpha(t) = \int_{\varepsilon_1 \times S^{n-1}} t^{\alpha+n/2}\, u\left(\sqrt{t}\,\rho\zeta, t\right)\left(\rho^2 - 1\right)^\alpha\, \rho^{n-1} d\rho\, d\zeta.$$

Here, ε_1 is the cycle represented by the unit circle centred at $+1$ and oriented in the negative direction, and the branch of $(\rho^2 - 1)^\alpha$ is chosen so that its value at the origin of ε_1 (which is also 0) is $\mathrm{e}^{\pi i\alpha}$.

Note that, for fixed t, the function

$$\rho \longmapsto \int_{S^{n-1}} u(\rho\zeta, t)\, d\zeta$$

is an even function which is holomorphic at the origin. Thus $(\rho, t) \mapsto \int_{S^{n-1}} u(\sqrt{t}\,\rho\zeta, t) d\zeta$ is a continuous function for $\rho \in \varepsilon_1$, and, in a neighbourhood of 0, and for fixed ρ in ε_1, this function is holomorphic in t at the origin. This proves that

$$v_\alpha(t) = \int_{\varepsilon_1 \times S^{n-1}} u\left(\sqrt{t}\,\rho\zeta, t\right)\left(\rho^2 - 1\right)^\alpha\, \rho^{n-1} d\rho\, d\zeta$$

is holomorphic at the origin. Furthermore, we have

$$\begin{aligned} v_\alpha(0) &= \frac{2\pi^{n/2}}{\Gamma(n/2)} u(0,0) \int_{\varepsilon_1} \left(\rho^2 - 1\right)^\alpha \rho^{n-1} d\rho \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} u(0,0) \mathrm{e}^{\pi i \alpha} \int_{\varepsilon_1} \left(1 - \rho^2\right)^\alpha \rho^{n-1} d\rho \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} u(0,0) \times i \sin \pi\alpha \, \frac{\Gamma(n/2)\,\Gamma(\alpha+1)}{\Gamma(1+\alpha+n/2)} \end{aligned}$$

$$v_\alpha(0) = -\frac{2i\pi^{1+n/2} u(0,0)}{\Gamma(-\alpha)\Gamma(1+\alpha+n/2)} \qquad \begin{array}{l} \text{if } -(\alpha+n/2) \notin \mathbb{N} \\ \text{and } \alpha \notin \mathbb{N}. \end{array}$$

In the case $\alpha \in \mathbb{N}$, the function $\rho \mapsto \int_{S^{n-1}} u(\sqrt{t}\rho\zeta, t)(1-\rho^2)^\alpha \rho^{n-1} d\zeta$ is holomorphic in a neighbourhood of the unit disk centred at 1, and therefore $g_\alpha(t) \equiv 0$.

On the other hand, if $-(\alpha + n/2) \in \mathbb{N}$, we can replace the integral $\int_{\varepsilon_1} (1-\rho^2)^\alpha \rho^{n-1} d\rho$ with an integral along a circle of arbitrarily large radius, which will tend to zero when the radius of this circle tends to infinity. Thus $v_\alpha(0) = 0$. In fact, one can show that 0 is a zero of v_α of order at least $-(\alpha + n/2)$, i.e., g_α is holomorphic at the origin.

Let us write the dependence of g_α on u explicitly. By differentiating under the integral sign,

$$g'_\alpha(u,t) = g_\alpha\left(\frac{\partial u}{\partial t}, t\right) - \alpha g_{\alpha-1}(u,t). \tag{1.4}$$

If we assume that $\partial u/\partial t, \partial^2 u/\partial t^2, \ldots$ satisfy the same decreasing conditions at infinity as u, then this formula shows that $g_{\alpha-1}$ is holomorphic at 0 if g_α is. Since $g_\alpha(t) = v_\alpha(t) t^{\alpha+n/2}$ is holomorphic for $\alpha + n/2 \in \mathbb{N} \cup \{0\}$, then $g_{-1-n/2}$, $g_{-2-n/2}, \ldots$ are also holomorphic at 0.

The behaviour of f_α near the origin

Let us choose γ_t^+ to be the semicircle of radius $\sqrt{t}$ centred at $+\sqrt{t}$ which is contained in the lower half plane, extended by the real half-axis $[2\sqrt{t}, \infty)$.

For the integral along the semicircle, we will use an upper bound which resembles the one given earlier. On the other hand, the integral along the half-axis $[2\sqrt{t}, \infty)$ can be written

$$\int_{2\sqrt{t}}^\infty \int_{S^{n-1}} u(\rho\zeta, t) \left(\rho^2 - t\right)^\alpha \rho^{n-1} d\zeta d\rho = \int_{2\sqrt{t}}^N \int_{S^{n-1}} + \int_N^\infty \int_{S^{n-1}} .$$

The integral $\int_N^\infty \int_{S^{n-1}}$ is obviously uniformly bounded for t sufficiently small, and for $\operatorname{Re}\alpha \geqslant -1$ we have

$$\left|\int_{2\sqrt{t}}^{N} \int_{S^{n-1}}\right| < \mathrm{Cst} \times \int_{4t}^{N^2} (\rho - t)^{-1}\, d\rho = \mathrm{Cst} \times \left[\log\left(N^2 - t\right) - \log 3t\right].$$

Hence

$$\lim_{t\to 0} t\, f_\alpha(t) = 0.$$

This result, along with the study of the behaviour of g_α shows that

$$\lim_{t\to 0} t\, k_\alpha(t) = 0 \quad \text{for} \quad \operatorname{Re}\alpha \geqslant -1, \tag{1.5}$$

and so

$$k_\alpha = \begin{cases} f_\alpha - \dfrac{g_\alpha}{\mathrm{e}^{2\pi i(\alpha+n/2)} - 1} & \text{if } \alpha + n/2 \notin \mathbb{Z} \\[2ex] f_\alpha - \dfrac{g_\alpha \log t}{2\pi i} & \text{if } \alpha + n/2 \in \mathbb{Z}. \end{cases}$$

Since we have already shown that k_α is a single-valued analytic function on $V - \{0\}$, condition (1.5) shows that k_α is in fact holomorphic at 0 for $\operatorname{Re}\alpha \geqslant -1$.

Finally, the fact that k_α is holomorphic at 0 *for all* α can be easily deduced from the formula for differentiating under an integral which resembles (1.4).

Remarks.

1) There are two parts to the above analysis. The first is geometric, i.e., the study of monodromy, and the second consists of finding upper bounds which lead to more precise formulae.

2) The example above is not so special. In fact, under the following general hypotheses: i) $X \to T$ is smooth, ii) ω is a relative multivalued differential form of maximal degree, such that its sheaf of determinations is of dimension $\mu = 1$, iii) $\pi|Y : Y \to T$ has a "*fold type singularity*" as in example 1), then we see without difficulty that the integral of ω has singularities of the same type as that of example 1).

The generality of example 1) is now clear, since singularities of "fold" type are generic (on the other hand, the restrictive hypothesis $\mu = 1$ was only made here to simplify the calculations. It would not be difficult to consider, more generally, a form of Nilsson type $\sum s^\alpha \log^p s\, dz$, where $s = z^2 - t$). Thus, example 1) suggests a programme to study "*typical generic integrals*".

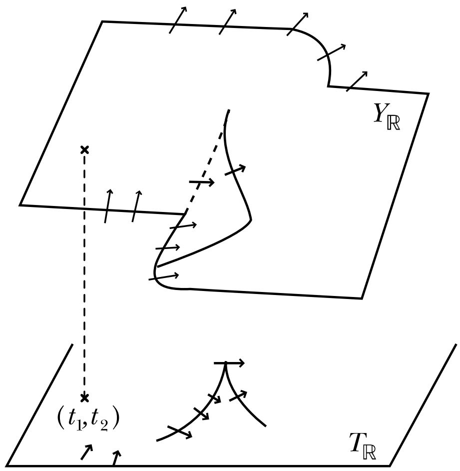
2 Second example

We will content ourselves with doing only the geometric part of the analysis of the integral of a form with a singularity of *"cuspidal" type*.

Let, for example,

$$f_\alpha(t) = \int_{\mathbb{R}} \left(x^3 + t_1 x + t_2\right)^\alpha dx.$$

As in example 1), we make sense of the integral by doing a complex displacement of $\mathbb{R}$ in every fibre above $(t_1, t_2) \in T = \mathbb{C}^2$.



For this, we take an infinitesimal displacement of $\mathbb{R}^3$ which is transverse to the real surface of singularities $Y_{\mathbb{R}} = \{(x, t_1, t_2) \in \mathbb{R}^3\,;\, x^3 + t_1 x + t_2 = 0\}$ and which is compatible with the projection map onto $T_{\mathbb{R}} = \mathbb{R}^2$. To each point of $T_{\mathbb{R}}$ outside the curve $4t_1^3 + 27t_2^2 = 0$, there corresponds a real point and two complex conjugate points of Y in the fibre over (t_1, t_2). It is clear that there are only two such displacements which give us two families of displacements of $\mathbb{R}^3$ in $\mathbb{C}^3$ which avoid the cusp, and are compatible with the projections onto $T_{\mathbb{R}}$. By integrating over the homology classes $h^-(t)$ obtained in this way in the fibre over $t = (t_1, t_2)$, we define two integrals $f_\alpha^\pm(t)$ (but, unlike what happened in example 1), these two integrals are not analytic continuations of each other).

To finish off, one would have to study the monodromy of $f_\alpha^+(t)$ and $f_\alpha^-(t)$. This monodromy is more difficult to study here because we have a representation of the (non-abelian) group $\pi_1(T^*, t_0)$ in $H_1(X_{t_0}^*, V)$ (where $T^*, X_{t_0}^*, V$ are defined as in chapter X).

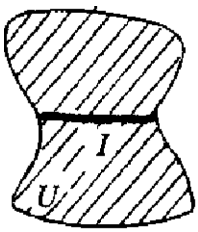
XIII

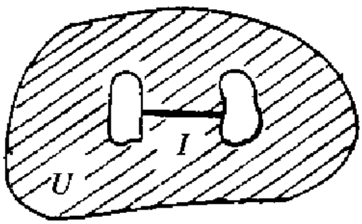
Hyperfunctions in one variable, hyperfunctions in the Nilsson class

Sato's theory of hyperfunctions is a natural setting for studying the boundary values of analytic functions. In the case of *a single variable*, this theory is very simple: here we will present what is necessary to study the boundary values of functions in the Nilsson class ("hyperfunctions in the Nilsson class", which, in reality, are just *distributions* of a very special type).

1 Definition of hyperfunctions in one variable

Let I be an open subset of $\mathbb{R}$. Then we define a *complex neighbourhood* of I to be an open subset U of $\mathbb{C}$ containing I, such that I is closed in U.





In the following, we will use the notation $\mathscr{O}(U)$ (resp. $\mathscr{M}(I)$) to denote the set of holomorphic functions on U (resp. complex-valued real analytic functions on I).

Let I be an open subset of $\mathbb{R}$, and let $f_1 \in \mathscr{O}(U_1 - I)$ and $f_2 \in \mathscr{O}(U_2 - I)$, where U_1 and U_2 are two complex neighbourhoods of I. Then we say that f_1 is *equivalent to* f_2 if $f_1 - f_2$ is the restriction of a *holomorphic* function on $U_1 \cap U_2$ to $(U_1 \cap U_2) - I$. It is clear that this is an equivalence relation.

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_13,

1.1 Definition. We define a hyperfunction on I to be any equivalence class with respect to this relation. If $f \in \mathscr{O}(U - I)$, the hyperfunction defined by f will be denoted by $[f]$. Let $\mathscr{B}(I)$ denote the set of all hyperfunctions on I.

Let $[f_1], [f_2] \in \mathscr{B}(I)$, $g \in \mathscr{M}(I)$ and let $\widehat{g} \in \mathscr{O}(U)$ be a complexification of g (i.e., $\widehat{g}|I = g$, where U is a complex neighbourhood of I). Then we easily see that $[f_1 + f_2]$ (resp. $[\widehat{g}\, f_1]$) only depends on $[f_1], [f_2]$ (resp. g and $[f_1]$): in this way, we can define the sum $[f_1] + [f_2]$ of two hyperfunctions, and the *product* $g[f_1]$ of a hyperfunction with an analytic function. These two operations define a $\mathscr{M}(I)$-module structure on $\mathscr{B}(I)$.

With these definitions, let $f \in \mathscr{O}(U - I)$. By replacing U with a smaller complex neighbourhood if necessary, we can assume that $U - I$ has *two connected components*, which enables us to write $f = f_+ - f_-$, where $f_+(z) = f(z)$ (resp. $f_-(z) = -f(z)$) in the part of U contained in the upper-half plane (resp. lower-half plane) and $f_+(z) = 0$ (resp. $f_-(z) = 0$) in the other part. Then $[f] = [f_+] - [f_-]$.

1.2 Definition. The hyperfunction $[f_+]$ (resp. $[f_-]$) is called the upper (resp. lower) boundary value of f.

Example. Let $\delta = -\frac{1}{2\pi i}[1/z]$ be the Dirac hyperfunction defined on an arbitrary open subset of $\mathbb{R}$ which contains 0. Then we have

$$\delta = -\frac{1}{2\pi i}\left(\frac{1}{x + i0} - \frac{1}{x - i0}\right),$$

where

$$\frac{1}{x + i0} = \left[\left(\frac{1}{z}\right)_+\right]$$

and

$$\frac{1}{x - i0} = \left[\left(\frac{1}{z}\right)_-\right].$$

Returning to the general case, the introduction of the hyperfunctions $[f_\pm]$ enables us to identify $\mathscr{M}(I)$ with a submodule of $\mathscr{B}(I)$. To see this, let $g \in \mathscr{M}(I)$ and let $\widehat{g} \in \mathscr{O}(U)$ be a complexification of g. By definition, we have $[\widehat{g}] = 0$, and hence $[\widehat{g}_+] = [\widehat{g}_-]$. In this way, $g \mapsto [g_+] = [g_-]$ defines an isomorphism from $\mathscr{M}(I)$ onto a submodule of $\mathscr{B}(I)$.

2 Differentiation of a hyperfunction

Let $[f]$ be a hyperfunction defined by $f \in \mathscr{O}(U - I)$. Then $[f']$ is a hyperfunction which only depends on $[f]$, which will be denoted by $[f]'$, and which will be called the *derivative of the hyperfunction* $[f]$.

Example. Let δ be the Dirac hyperfunction defined above. We have:

$$\delta' = \frac{1}{2\pi i} \left[\frac{1}{z^2}\right].$$

More generally,

$$\delta^{(n)} = \frac{(-1)^{n+1}}{2\pi i n!} \left[\frac{1}{z^{(n+1)}}\right].$$

3 The local nature of the notion of a hyperfunction

Let $I' \subset I$ be two open subsets of $\mathbb{R}$, and let $[f] \in \mathscr{B}(I)$ be defined by $f \in \mathscr{O}(U - I)$. Since $U' = U - (I - I')$ is a complex neighbourhood of I', $[f|U' - I']$ is a hyperfunction on I' which we call the *restriction* of $[f]$ to I', and which we denote by $[f]|I'$.

3.1 Definition. Let $I = \bigcup_\alpha I_\alpha$, where $(I_\alpha)_\alpha$ is a family of open subsets of $\mathbb{R}$. A *local hyperfunction* on I is *a family* $(h_\alpha)_\alpha$, where $h_\alpha \in \mathscr{B}(I_\alpha)$ such that

$$\forall\, \alpha, \beta \qquad h_\alpha | I_\alpha \cap I_\beta = h_\beta | I_\alpha \cap I_\beta. \tag{3.2}$$

3.3 Theorem. *Suppose we are given a local hyperfunction $(h_\alpha)_\alpha$ on I. Then there exists one and only one hyperfunction $h \in \mathscr{B}(I)$ such that for all α, $h|I_\alpha = h_\alpha$. In other words, hyperfunctions form a sheaf on $\mathbb{R}$ which we call the* sheaf of hyperfunctions on $\mathbb{R}$.

Proof. The proof of theorem 3.3 is based on a classical theorem due to Cousin, which we state here without proof:

3.4 Theorem. *Let $(U_\alpha)_\alpha$ be a family of open subsets of $\mathbb{C}$, and for each α, β suppose that there exists a function $f_{\alpha\beta}$ which is holomorphic on $U_\alpha \cap U_\beta$ such that we have*

$$\forall\, \alpha, \beta \qquad f_{\alpha\beta} + f_{\beta\alpha} = 0 \quad \textit{on} \quad U_\alpha \cap U_\beta; \tag{3.5}$$

$$\forall\, \alpha, \beta, \gamma \qquad f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \quad \textit{on} \quad U_\alpha \cap U_\beta \cap U_\gamma. \tag{3.6}$$

Then there exists a family $(f_\alpha)_\alpha$ such that $\forall\, \alpha, f_\alpha \in \mathscr{O}(U_\alpha)$ and $\forall\, \alpha, \beta,\ f_{\alpha\beta} = f_\alpha - f_\beta$ on $U_\alpha \cap U_\beta$.

The proof of theorem 3.3 can be immediately deduced from theorem 3.4, as follows. Let $f_\alpha \in \mathscr{O}(U_\alpha - I_\alpha)$, and $f_\beta \in \mathscr{O}(U_\beta - I_\beta)$ be the representatives of the classes which define the hyperfunctions h_α and h_β respectively. Then condition (3.2) can be expressed as:

$$f_\alpha - f_\beta = f_{\alpha\beta},$$

where $f_{\alpha\beta}$ is the restriction to $(U_\alpha \cap U_\beta) - (I_\alpha \cap I_\beta)$ of a holomorphic function on $U_\alpha \cap U_\beta$ which we will denote from now on by $f_{\alpha\beta}$.

We therefore have a family of functions $(f_{\alpha\beta})$ which are holomorphic on $U_\alpha \cap U_\beta$ and which certainly verify conditions (3.5) and (3.6). By Cousin's theorem, there exists a family $(g_\alpha)_\alpha$ such that $g_\alpha \in \mathscr{O}(U_\alpha)$, and:

$$\forall\, \alpha, \beta \qquad f_{\alpha\beta} = g_\alpha - g_\beta \quad \text{on} \quad U_\alpha \cap U_\beta.$$

But then the function $f_\alpha - g_\alpha$ in turn defines h_α, and we have:

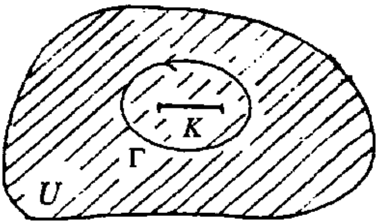
$$\forall\, \alpha, \beta \qquad f_\alpha - g_\alpha = f_\beta - g_\beta \quad \text{on} \quad (U_\alpha \cap U_\beta) - (I_\alpha \cap I_\beta).$$

This shows that the family $(f_\alpha - g_\alpha)$ can be extended to give a function $f \in \mathscr{O}\left(\bigcup_\alpha U_\alpha - \bigcup_\alpha I_\alpha\right)$. □

4 The integral of a hyperfunction

Let h be a hyperfunction on I with *compact support* K, i.e., $h = [f]$, where f is holomorphic on $U - K$, and U is a complex neighbourhood of I.

Let Γ be a closed curve in $U - K$ which surrounds K. Then Cauchy's theorem shows that the integral $-\oint_\Gamma f(z)\, dz$ does not depend on Γ. By Cauchy's theorem once again, this integral only depends on $h = [f]$. It is this integral, denoted $\int h$, which we will call the *integral of the hyperfunction h*.



Example. Let us calculate the integral $\int g(x)\delta^n(x)dx$, where $g \in \mathscr{M}(\mathbb{R})$. Let $\widehat{g}$ be holomorphic in a complex neighbourhood of $\mathbb{R}$ such that $\widehat{g}|\mathbb{R} = g$. Then

$$g(x)\delta^{(n)}(x) = \frac{(-1)^{n+1}}{2\pi i n!}\left[\widehat{g}(z)\frac{1}{z^{n+1}}\right]$$

is a hyperfunction with support $\{0\}$. Cauchy's integral formula gives

$$\int g(x)\delta^{(n)}(x)dx = \frac{(-1)^n}{2\pi i n!}\oint_{S^1}\frac{\widehat{g}(z)dz}{z^{n+1}} = g^{(n)}(0),$$

which subsequently enables us to identify $\delta^{(n)}(x)$ with the n-th derivative of the Dirac measure concentrated at 0.

5 Hyperfunctions whose support is reduced to a point

Consider, for example, a hyperfunction h with support $\{0\}$. Let $f \in \mathscr{O}(U - \{0\})$ which defines h.

Let $f(z) = \sum_0^\infty \frac{a_n}{z^{n+1}} + g(z)$ be the expansion of f in a neighbourhood of the origin, where $g(z)$ is a holomorphic function. Then

$$h = [f] = \sum_0^\infty 2\pi i(-1)^{n+1}\, n!\, a_n\, \delta^{(n)}(x).$$

We therefore see that h is not a distribution in general, because every distribution whose support is reduced to a point is equal to a *finite sum* of derivatives of δ.

6 Hyperfunctions in the Nilsson class

Recall that a function in the Nilsson class on an open subset U of $\mathbb{C}$ is a multivalued analytic function of finite determination on $X - Y$, with moderate growth near Y, where Y is a set of isolated points of U.

We have already proved that f is in the Nilsson class on U if and only if it is a solution of a differential equation which has only regular singular points on U.

6.1 Definition. $h = [f]$ is a *hyperfunction in the Nilsson class* on an open subset I of $\mathbb{R}$ if $f \in \mathscr{O}(U - I)$ is a branch of a function in the Nilsson class on U.

Note that this is a local definition: the set of hyperfunctions in the Nilsson class forms a subsheaf (a sheaf of submodules) of $\mathscr{B}$, which we denote by $\mathscr{B}_{\mathrm{Nils}}$.

6.2 Theorem. *$h \in \mathscr{B}_{\mathrm{Nils}}(I)$ if and only if h is a solution of a differential equation on I with regular singular points, i.e.,*

$$\frac{d^\mu}{dx^\mu}h + a_1(x)\frac{d^{\mu-1}}{dx^{\mu-1}}h + \cdots + a_\mu(x)h = 0, \tag{6.3}$$

where each a_i is a meromorphic function having only poles of order $\leqslant i$, or, equivalently, if and only if h is a solution of a differential equation of the form

$$\left(x\frac{d}{dx}\right)^\mu h + b_1(x)\left(x\frac{d}{dx}\right)^{\mu-1} h + \cdots + b_\mu(x)h = 0, \tag{6.4}$$

where $b_1 \in \mathscr{M}(I)$.

Proof. It is easy to prove that h is a solution of (6.3) if and only if it is a solution of (6.4), where the b_i are suitably chosen.

Now let us show that these conditions are equivalent to $h \in \mathscr{B}_{\mathrm{Nils}}(I)$. Since our conditions are obviously local, it is enough to consider just one singular point which we can assume is situated at the origin.

In this case, the condition is obviously necessary. Conversely, let us suppose that $h = [f]$ is a solution of $P(xd/dx)h = 0$, where P is a polynomial with coefficients in $\mathscr{M}(I)$. By definition of the derivative,

$$[P\,(zd/dz)\,f] = P\left(x\frac{d}{dx}\right)h = 0,$$

and therefore $P(zd/dz)f = g \in \mathscr{O}(U)$, where U is a neighbourhood of 0 in $\mathbb{C}$. Then g can be written:

$$g(z) = z^k u(z), \quad u \in \mathscr{O}(U), \quad u(0) \neq 0.$$

Now g is the solution to

$$\left(z\frac{d}{dz} - b(z)\right)g = 0,$$

where $b = k + u'/u$ is holomorphic in a neighbourhood of 0. Therefore f is a solution of

$$Q\,(zd/dz)\,f \equiv \left(z\frac{d}{dz} - b(z)\right)P\,(zd/dz)\,f = 0$$

which proves that f is an analytic function in the Nilsson class. □

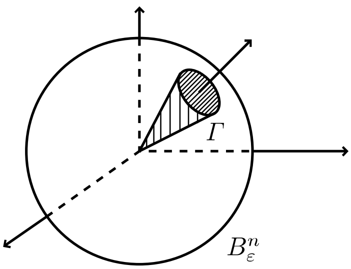
We see by the above, and by the expansion of a multivalued function in the Nilsson class in terms of powers of logarithms, that $\mathscr{B}_{\mathrm{Nils}}(I)$ is a submodule of $\mathscr{B}(I)$ spanned by the hyperfunctions $(x \pm i0)^\alpha = [z_\pm^\alpha]$, $\log^p(x \pm i0) = [(\log^p z)_\pm]$ (and their transforms under translations). Hyperfunctions in the Nilsson class therefore form a very special class of *distributions* on I.

XIV

Introduction to Sato's microlocal analysis

1 Functions analytic at a point x and in a direction

Let $x \in \mathbb{R}^n$, and let v be a direction in $\mathbb{R}^n$. Let us consider the set of all holomorphic functions on open subsets of $\mathbb{C}^n$ of the form $U + i\Gamma$, where U is an open neighbourhood of x in $\mathbb{R}^n$, and Γ is the intersection of the ball B^n_ε centred at the origin with radius ε, with an *open convex cone* with apex O and containing the direction v.



Let $f_1 \in \mathscr{O}(U_1 + i\Gamma_1)$ and $f_2 \in \mathscr{O}(U_2 + i\Gamma_2)$ be two such functions. Then we say that f_1 *is equivalent to* f_2 if they are equal in the intersection $(U_1 + i\Gamma_1) \cap (U_2 + i\Gamma_2)$. Each equivalence class for this relation is, by definition, a function which is *analytic at x in the direction v*.

The functions which are analytic at points x of $\mathbb{R}^n$ in the directions v are therefore kinds of "germs" defined not on $\mathbb{R}^n$ but on the *fibre bundle of directions tangent to* $\mathbb{R}^n$. Note that the open sets $U + i\Gamma$ do not contain x!

2 Functions analytic in a field of directions on $\mathbb{R}^n$

Let v be a field of directions on $\mathbb{R}^n$ which are defined on an open subset U of $\mathbb{R}^n$. Then we define a *function which is analytic in the field of directions v* to be a function f which is holomorphic on an open subset of $\mathbb{C}^n$ of the form

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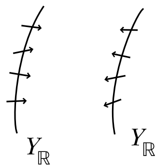
Universitext, DOI 10.1007/978-0-85729-603-0_14,

$\{x + iy\,;\, x \in U,\, y \in \Gamma_x\}$, where each Γ_x is the intersection of a ball B^n_ε with a convex open cone with apex O, containing the direction v_x.[1]

Example. Let $Y_{\mathbb{R}}$ be a *smooth* hypersurface in $\mathbb{R}^n$ defined by an equation of the form $s(x_1, \ldots, x_n) = 0$, where s is an analytic function on $\mathbb{R}^n$ with real coefficients. Let Y be the complexified hypersurface of $Y_{\mathbb{R}}$ which is contained in a complex neighbourhood U of $\mathbb{R}^n$. Let v be a field of directions which are *transverse* to $Y_{\mathbb{R}}$. Observe that the function $[s(z)]^\alpha$ ($\alpha \in \mathbb{C}$) is analytic in the field of directions v.

By the above, we can in effect find a family of sufficiently small displacements of $\mathbb{R}^n$ which avoid Y, whose associated infinitesimal displacement is v. This means that we can find an open subset of $\mathbb{C}^n$ corresponding to the field of directions v in the way indicated earlier, such that $s(z)$ does not vanish on this open set; then $[s(z)]^\alpha$ is defined and holomorphic on it.

Note that, from the point of view of "boundary values" (cf. §3 hereafter), there is no point in distinguishing between: i) the function $[s(z)]^\alpha$ considered as an analytic function in a field of directions v transverse to $Y_{\mathbb{R}}$, and ii) the same function, considered as analytic in every other field of directions which are transverse to $Y_{\mathbb{R}}$ in the same direction. However, it is absolutely essential to make a distinction from $[s(z)]^\alpha$ considered as an analytic function in the field of directions which are *opposite to* v. This idea will appear more clearly in §4.



Let us try to apply the preceding notations to the singular integrals which we have already considered (Chap. XII).

1) In the first example, we showed that the integral of a function which is analytic in a field of directions v transverse to the paraboloid $x^2 - t = 0$ is a function which is analytic in the field of directions given by the projection of v onto $T_{\mathbb{R}} = \mathbb{R}$ (see Fig. XIV.1).

This means, moreover, that the integral is a function which is analytic in the half-neighbourhood above the open interval I in $\mathbb{R}$ (see Fig. XIV.2). Likewise, by considering the field of opposite directions to v, we have a function which is analytic in a half-neighbourhood below I (see Fig. XIV.3).

2) In the second integral, the integrand is analytic in the two opposite fields of directions which are transverse to the surface $x^3 + t_1 x + t_2 = 0$.

[1] Here, again, it would be more correct to define "a function which is analytic in a field of directions" to be an equivalence class defined as in §1.

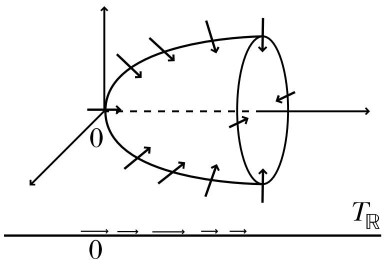


Fig. XIV.1.

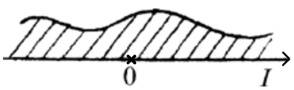


Fig. XIV.2.

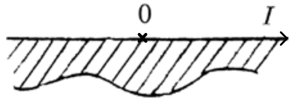


Fig. XIV.3.

Thus the integral is analytic in the two opposite fields of directions which are "transverse" to the curve $4t_1^3 + 27t_2^2 = 0$ in $\mathbb{R}^2$ (see Fig. XIV.4).

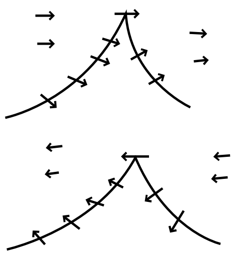


Fig. XIV.4.

Exercise. Check that there does not exist a complex displacement of $\mathbb{R}^2$ which avoids the complex points of the curve $4t_1^3 + 27t_2^2 = 0$. How can you reconcile this with the results on the analyticity of the second integral in chapter XII?

3 Boundary values of a function which is analytic in a field of directions

Using the notions introduced above, we can define the notion of the *boundary value* of a function which is analytic in a field of directions. One can either work with *distributions* or *hyperfunctions*, according to one's personal taste.

3.1 Boundary values in the sense of distributions

Let f be a function which is analytic in a field of directions v on $\mathbb{R}^n$. Consider the distribution defined by

$$\langle T, \varphi \rangle = \lim_{\lambda \to 0} \int_{\mathbb{R}^n} f\left(x + i y_\lambda(x)\right) \varphi(x) dx, \quad \forall \varphi \in C_c^\infty\left(\mathbb{R}^n\right),$$

where $y_\lambda(x) = \lambda v_x$, $\lambda > 0$.

In other words, $\langle T, \varphi \rangle$ is the limit of integrals in the space $\mathbb{R}^n$ which has been slightly displaced in the direction v.

Here it is necessary to assume a "moderate growth" condition for $f(x+iy)$ as $\|y\| \to 0$ in order for the limit to exist and define a distribution, which is called the *boundary value of f in the direction v*.

3.2 Boundary values in the sense of hyperfunctions

This theory is more formal. The notion of a hyperfunction is defined using a sum of "formal" boundary values, with a rather sophisticated equivalence relation which we will not write down explicitly here (this definition, in the case $n > 1$, can only be introduced in a natural way using the language of cohomology).

Recall the:

3.3 Theorem. *Every distribution on $\mathbb{R}^n$ can be written as a sum of boundary values of functions which are analytic in at most $n+1$ fields of directions (we can fix from the outset any $n+1$ directions whose convex hull is the entire space).*

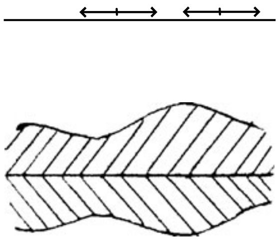
Remark. The decomposition given by theorem 3.3 is not unique. Consider, for example, a distribution which is invariant under rotations around the origin – say, the Dirac distribution concentrated at the origin. Then, starting from any decomposition as boundary values, we can obtain other ones by rotating the $n+1$ corresponding fields around the origin.

Examples.

1) In the case $n = 1$, the theorem simply says that every distribution is the sum of two boundary values of functions which are analytic in the two opposite fields of directions, i.e., the sum of boundary values of two functions which are analytic in two half-neighbourhoods of $\mathbb{R}$ in $\mathbb{C}$ respectively.

2) *The distribution $\delta(s)$*: Let $1/(s+i0)$, $1/(s-i0)$ denote the two analytic functions defined by $1/s$ in two fields of opposite directions which are transverse to the hypersurface $s = 0$ (see the example of §2, with $\alpha = -1$). Then the distribution $\delta(s)$ is defined by

$$\delta(s) = -\frac{1}{2\pi i}\left(\frac{1}{s+i0} - \frac{1}{s-i0}\right). \tag{3.4}$$



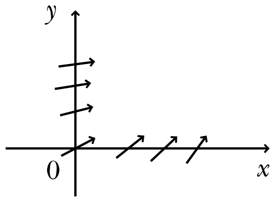
We therefore obtain a decomposition of $\delta(s)$ as a sum of only *two* boundary values.

3) Consider the distribution $\delta(x, y) = \delta(x)\delta(y)$ on $\mathbb{R}^2$, where $\delta(x)$ is the Dirac distribution of $\mathbb{R}$ concentrated at x.

By regarding $\delta(x)$ and $\delta(y)$ as distributions on $\mathbb{R}^2$ associated to the lines $x = 0$ and $y = 0$ respectively, we have, by example 2), a decomposition of $\delta(x, y)$ of the form

$$\delta(x, y) = \sum \frac{\pm 1}{2\pi i} \left(-\frac{1}{x \pm i0} \cdot \frac{1}{y \pm i0} \right) \tag{3.5}$$

where each term can be defined by a field of directions contained in one of the four quadrants of $\mathbb{R}^2$.



Note that in this way we obtain a decomposition of δ as a sum of four boundary values, one more than the number of boundary values required for the decomposition given by theorem 3.3.

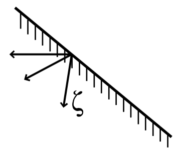
In the following, we will use the word "hyperfunction", but the reader who is intimidated by this word can equally well replace it by "distribution".

In any case, what we are interested in here is not the notion of hyperfunctions as a *generalization* of the notion of distributions, but rather *Sato's microlocal analysis*, which is much more naturally placed in the context of the theory of hyperfunctions.

4 The microsingular support of a hyperfunction (or spectral support, or essential support, or singular spectrum, or wave front set, etc.)

In the sequel, a *cotangent* direction (or a *codirection*) to $\mathbb{R}^n$ will be an equation of a hyperplane in the tangent space modulo a positive constant factor. This comes to the same thing as saying that a codirection in $\mathbb{R}^n$ is determined by a half-space in the tangent space of $\mathbb{R}^n$.

4.1 Definition. i) Let h be a hyperfunction on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and let ζ be a codirection at x. Then we say that h is *microanalytic at the point x and in the codirection ζ*, if h can be written as a sum of boundary values of functions which are analytic in all the *directions* which are situated *on the wrong side* of ζ (i.e., situated in the half-space opposite to the one defined by ζ).



ii) The set of all codirections where h is not microanalytic is called the *microsingular support* of h.

The microsingular support of a hyperfunction is therefore a subset of the fibre bundle of codirections of $\mathbb{R}^n$.

Examples.

1) Let $(s+i0)^\alpha$ be the boundary value defined by a function s which is analytic in a field of directions which are transverse to the smooth hypersurface $Y_\mathbb{R} = \{x \in \mathbb{R}^n\,;\, s = 0\}$, where the directions point in the direction $ds > 0$.

- If $x \notin Y_\mathbb{R}$ then the function is analytic, and therefore microanalytic in every codirection. In other words, the microsingular support is $\varnothing$.
- If $x \in Y_\mathbb{R}$, then there is only one bad codirection, namely the one corresponding to the half-space $ds > 0$.

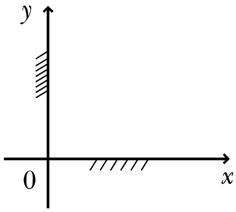
In conclusion, the microsingular support is the field of codirections defined by the tangent half-spaces $ds > 0$.

In the same way, we find that the microsingular support of $(s-i0)^\alpha$ is the field of codirections defined by the tangent half-spaces $ds < 0$.

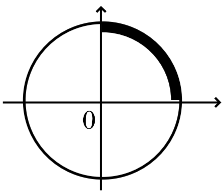
The microsingular support of the Dirac hyperfunction $\delta(s)$ is therefore the union of these two fields of codirections.

2) Now consider the hyperfunction $\delta(x,y) = \delta(x)\delta(y)$ on $\mathbb{R}^2$. We can calculate the microsingular support of δ by calculating the microsingular support of each term in the decomposition (2) above. Consider, for example, the term $\frac{1}{x+i0} \cdot \frac{1}{y+i0}$:

- if $(x, y) \notin (Ox) \cup (Oy)$, then the microsingular support is $\varnothing$;



- if $(x, y) \in Ox$, $x \neq 0$, then there is only one microsingular codirection, namely the one which points upwards;
- likewise, if $(x, y) \in Oy$ and $y \neq 0$, the only microsingular codirection is the one which points to the right;
- now let $(x, y) = (0, 0)$. Then the microsingular codirections are those whose *correct side* contains the first quadrant. In other words, the microsingular support of $\frac{1}{x+i0} \cdot \frac{1}{y+i0}$ at the origin is certainly included in the first quadrant of the unit circle.



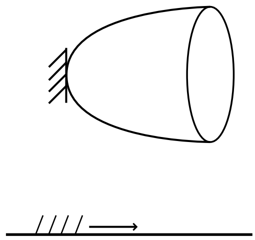
In the same way, the microsingular supports of the three other terms in the decomposition (3.3) are contained in the three other quadrants of the unit circle. In fact, one can show that these microsingular supports are exactly these quarter-circles. Since the hyperfunction δ is invariant under rotations around the origin, its microsingular support at the origin is the whole circle, which is the union of these four quarter-circles (in fact, its invariance under rotations shows that its microsingular support is either empty, or equal to the whole circle, but if it were empty, the "edge of the wedge" theorem would tell us that δ is analytic at the point O, which is absurd).

5 The microsingular support of an integral

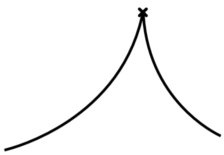
Can one say something about the microsingular support of an integral when we know the microsingular support of the integrand?

First of all, let us return to the examples of chapter XII.

1) In example 1), the microsingular support of the integrand is the field of codirections determined by the half-spaces which are tangent to the paraboloid $x^2 - t = 0$. On the other hand, the integral has just one microsingular codirection, namely the codirection at the origin which is the projection of the microsingular codirection of the integrand at 0.



2) Likewise, in example 2), the integral has only one microsingular codirection, namely the one at the singular point of the curve $4t_1^3 + 27t_2^2 = 0$, which is the projection of the microsingular codirection of the integrand at the "cusp" of the hypersurface $x^3 + t_1 x + t_2 = 0$.



The situation described in these two examples is in fact very general:

5.1 Theorem. *Let $X_{\mathbb{R}} \xrightarrow{\pi} T_{\mathbb{R}}$ be a proper smooth map between real analytic manifolds, and let ω be a relative differential form of maximal degree whose coefficients are hyperfunctions.*

Then there is a notion of the integral of ω along the fibres of π, *which gives a hyperfunction on $T_{\mathbb{R}}$ whose microsingular support has the following property:*

A codirection ζ at a point $t \in T_{\mathbb{R}}$ can only be microsingular for the integral if there exists a point x in the fibre above t where the inverse image $\pi_x^*(\zeta)$ is a microsingular codirection for the integrand.

The results and examples below suggest the following problem:

Problem. Generalize the notion of a hyperfunction in the Nilsson class (defined in Chap. XIII for the case of a single variable) to the case of an arbitrary number of variables.

Can one formulate a "*Nilsson theorem for hyperfunctions*", i.e., show that, under certain conditions to be specified, an integral of the kind considered in theorem 5.1 is in the Nilsson class if its integrand is in the Nilsson class?

This problem seems to have connections with many other problems of great interest, and suggests numerous directions for future research. I will limit myself to mentioning just two examples by way of conclusion.

5.2 Oscillatory integrals

In many problems in mathematical physics, one encounters "oscillatory integrals" of the form

$$\widetilde{f}(k) = \int_{\mathbb{R}^n} \mathrm{e}^{iks(x_1,\ldots,x_n)}\varphi(x_1,\ldots,x_n)dx_1\cdots dx_n,$$

and one must study their asymptotic behaviour as $k \to \infty$. By writing

$$\widetilde{f}(k) = \int_{\mathbb{R}^{n+1}} \mathrm{e}^{ikt}\delta(t - s(x_1,\ldots,x_n))\varphi(x_1,\ldots,x_n)dx_1\cdots dx_n,$$

we see that we can consider $\widetilde{f}(k)$ as the Fourier transform of the function

$$f(t) = \int_{\mathbb{R}^n} \delta(t - s(x_1,\ldots,x_n))\varphi(x_1,\ldots,x_n)dx_1\cdots dx_n.$$

Let us consider, for the sake of argument, the case where $s = x_1^2 + \cdots + x_n^2$. By writing

$$\delta(t-s) = \frac{1}{2\pi i}\left(-\frac{1}{t-s+i0} - \frac{1}{t-s-i0}\right),$$

we see that $f(t)$ is the difference between two integrals of the kind studied in chapter XII (namely $f_\alpha^-(t) - f_\alpha^+(t)$, where $\alpha = -1$). We therefore have $f(t) \sim [t^{n/2} - 1]_+$, which gives another proof of the classical stationary phase formula

$$\widetilde{f}(t) \sim k^{-n/2}.$$

Suggested reading

a) B. Malgrange – "Intégrales asymptotiques et monodromie", *Ann. scient. Éc. Norm. Sup.* 4e *série* (1974).
b) V. Arnold – *Uspekhi Mat. Nauk* **5** (1973), English trans. in *Russian Math. Surveys* (1973).

5.3 The behaviour of light waves in the neighbourhood of caustics

This problem, which is attracting the attention of mathematicians once again, is closely connected to the previous one.

Suggested reading

a) The classical physics textbooks on wave optics and the approximation of geometric optics.

b) Duistermaat – "Oscillatory integrals, Lagrange immersions and unfolding of singularities", *Comm. Pure Appl. Math.* (1974).

A

Construction of the homology sheaf of X over T

Proposition 1. *Let $\pi : X \to T$ be a smooth map of real manifolds, and let $\dim T = r$. For every subset P of T, let $X_P = \pi^{-1}(P)$. Then for every point t of T, we have the following isomorphisms of homology groups (with integer coefficients):*

$$H_p(X_t) \approx H_{p+r}\left(X, X_{T-\{t\}}\right) \approx \varinjlim_{U \to t} H_{p+r}\left(X, X_{T-U}\right), \quad (\forall\, p \in \mathbb{N}),$$

where the inductive limit is taken over the direct family of open neighbourhoods U of t.

Proof.

a) Lemma. *Let (A, B) be a pair of topological spaces, let $\{A_i\}$ be an increasing sequence of open subsets of A such that $\bigcup A_i = A$, and let $\{B_i\}$ be an increasing sequence of open subsets of B, such that $\bigcup B_i = B$. Then $H_*(A, B) = \varinjlim H_*(A_i, B_i)$.*

Proof. This is already true for groups of chains. □

b) First step: by applying the lemma to the pairs $(X, X_{T-U}) \subset (X, X_{T-\{t\}})$, we obtain the second isomorphism (this does not use the assumption that π is smooth).

Second step: the smoothness of π proves that each fibre X_t of π is a submanifold of X, and we therefore have an increasing sequence of relatively compact open subsets K_i of X_t such that $X_t = \bigcup K_i$. By applying the lemma to the pairs $(K_i, \varnothing) \subset (X_t, \varnothing)$, we deduce that $H_*(X_t) \approx \varinjlim H_*(K_i)$.

Third step: the smoothness of π proves that every relatively compact open subset K_i of X_t has a neighbourhood L_i in X such that $\pi|L_i$ makes L_i into a *trivial fibre bundle* with fibres K_i and with base a small ball $U_i \ni t$, and furthermore, such that $i < j \Rightarrow U_j \subset U_i$ and $L_i|_{U_j} \subset L_j$ (see Fig. A.1).

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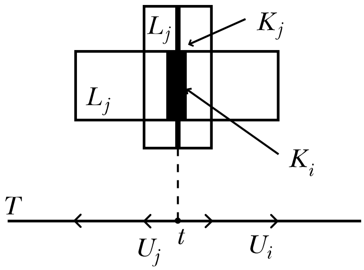

Fig. A.1.

By applying the lemma to the pairs $(L_i \cup X_{T-U_i'}, X_{T-U_i'}) \subset (X, X_{T-\{t\}})$, where U_i' is a ball which is a little smaller than U_i, we obtain the isomorphism:

$$H_* \left(X, X_{T-\{t\}}\right) \approx \varinjlim H_* \left(L_i \cup X_{T-U_i'}, X_{T-U_i'}\right).$$

Now, by excision, we have the isomorphism:

$$H_* \left(L_i \cup X_{T-U_i'}, X_{T-U_i'}\right) \approx H_* \left(L_i, L_i \cap \pi^{-1} \left(U_i - U_i'\right)\right),$$

and because L_i is a direct product, we have:

$$H_* \left(L_i, L_i \cap \pi^{-1} \left(U_i - U_i'\right)\right) \approx H_* \left(K_i \times U_i, K_i \times \left(U_i - U_i'\right)\right);$$

but since $(U_i, U_i - U_i') \sim (B_r, S_{r-1})$ (B_r is a ball of dimension r, S_{r-1} its boundary), we have by Kunneth's theorem the isomorphisms:

$$\begin{aligned} H_{p+r} \left(K_i \times U_i, K_i \times \left(U_i - U_i'\right)\right) &\approx H_p \left(K_i\right) \otimes H_r \left(B_r, S_{r-1}\right) \\ &\approx H_p \left(K_i\right) \otimes \mathbb{Z} \approx H_p \left(K_i\right). \end{aligned}$$

Therefore,

$$H_{p+r} \left(X, X_{T-\{t\}}\right) \approx \varinjlim H_p \left(K_i\right).$$

By the second step, we finally obtain the first isomorphism of the proposition:

$$H_{p+r} \left(X, X_{T-\{t\}}\right) \approx H_p \left(X_t\right). \qquad \square$$

Example. $X = S^1 \times \mathbb{R}, T = \mathbb{R}, \pi : X \to T$ is the canonical projection:

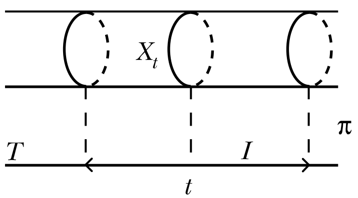

Fig. A.2.

X_t is a circle; let σ be the cycle in X_t represented by this circle; let I be an interval in T, $t \in I$; then we can consider $\sigma \times I$ as a cycle in $(X, X_{T-\{t\}})$, and the isomorphism $H_1(X_t) \approx H_2(X, X_{T-\{t\}})$ is defined by $[\sigma] \mapsto [\sigma \times I]$.

The homology sheaf of X over T

Let $\pi : X \to T$ be a smooth map of real manifolds, with dim $T = r$. Then, for every $p \in \mathbb{N}$, one can define a sheaf of groups on T which is called the homology sheaf in degree p of X over T, and which is denoted by $H_p(X/T)$: it is the sheaf associated to the following presheaf:

$$U(\text{open subset of } T) \rightsquigarrow H_{p+r}(X, X_{T-U}).$$

By the previous proposition, the fibre of this sheaf over the point t of T is $H_p(X_t)$.

Definition. A homology class of degree p on X_t which depends continuously on t is a section of the homology sheaf of X over T in degree p.

Remark. In our setting, where $\pi : X \to T$ is a smooth map of complex analytic manifolds of real dim $T = r = 2k$, the homology sheaf of X over T in degree p is the sheaf associated to the presheaf:

$$U(\text{open subset of } T) \rightsquigarrow H_{p+2k}(X, X_{T-U});$$

the fibre of this sheaf over $t \in T$ is still $H_p(X_t)$.

B

Homology groups with local coefficients

Let X be a topological space, and let $\mathscr{V}$ be a locally constant sheaf of abelian groups on X.

A (singular) chain of degree p in X with values in the local system of coefficients $\mathscr{V}$ is, by definition, a finite formal sum $\sum a_i s_i$, in which $s_i : \Delta_p \to X$ is a singular simplex of X, a_i is a section of the sheaf $s_i^{-1}(\mathscr{V})$ on Δ_p (where $s_i^{-1}(\mathscr{V})$ is the inverse image sheaf of $\mathscr{V}$ under the map s_i). These chains form an abelian group which we denote by $C_p(X, \mathscr{V})$ ($p \in \mathbb{N}$).

Let $\pi : X' \to X$ be a continuous map of topological spaces, and let $\mathscr{V}' = \pi^{-1}(\mathscr{V})$ be the inverse image sheaf of $\mathscr{V}$ under π. Then we have a homomorphism of groups $\widehat{\pi} : C_p(X', \mathscr{V}') \to C_p(X, \mathscr{V})$ defined by: $\sum a_i s'_i \to \sum a_i \pi s'_i = \sum a_i s_i$.

Remark. In the left-hand expression, a_i is a section of the sheaf $s_i'^{-1}(\mathscr{V}')$, and a_i in the right-hand expression is a section of the sheaf $s_i^{-1}(\mathscr{V})$, but it is clear that $s_i'^{-1}(\mathscr{V}') = s_i'^{-1}(\pi^{-1}(\mathscr{V})) = (\pi s'_i)^{-1}(\mathscr{V}) = s_i^{-1}(\mathscr{V})$.

Let $d_j : \Delta_{p-1} \to \Delta_p$ $(j = 0, 1, \ldots, p)$ be the face maps in the definition of the boundary of the standard simplex. Then we can define the "boundary" map:

$$\begin{aligned} d : \; C_p(X, \mathscr{V}) &\longrightarrow C_{p-1}(X, \mathscr{V}) \\ \sum_i a_i s_i &\longmapsto \sum_{j=0}^{p} (-1)^j \sum_i d_j^* a_i \, (s_i \circ d_j) \, . \end{aligned}$$

One verifies immediately that $d^2 = 0$, and as usual we define:

$$H_p(X, \mathscr{V}) = \frac{\operatorname{Ker}\Big(C_p(X, \mathscr{V}) \xrightarrow{d} C_{p-1}(X, \mathscr{V})\Big)}{\operatorname{Im}\Big(C_{p+1}(X, \mathscr{V}) \xrightarrow{d} C_p(X, \mathscr{V})\Big)} .$$

This is the homology group of X with values in the local system of coefficients $\mathscr{V}$, in degree p.

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Let $\pi : X' \to X$ be a continuous map of topological spaces. One can check that d and $\widehat{\pi}$ commute, and thus π induces a homomorphism of groups:

$$H_p(X', \mathscr{V}') \longrightarrow H_p(X, \mathscr{V}).$$

If $\tau : \mathscr{V} \to \mathscr{W}$ is a homomorphism of locally constant sheaves of abelian groups on X, we have an obvious homomorphism defined by:

$$\begin{aligned} C_p(X, \mathscr{V}) &\longrightarrow C_p(X, \mathscr{W}) \\ \sum a_i s_i &\longmapsto \sum \alpha_i s_i, \end{aligned}$$

where $\alpha_i = s_i^{-1}(\tau)(a_i)$ is a section of $s_i^{-1}(\mathscr{W})$ (see the figure). This homomorphism commutes with d, and by taking the quotient, we obtain a homomorphism of groups: $H_p(X, \mathscr{V}) \to H_p(X, \mathscr{W})$.

$$\begin{tikzcd}
s_i^{-1}(\mathscr{V}) \arrow[dr] \arrow[dd, "s_i^{-1}(\tau)"'] & & & \mathscr{V} \arrow[dl] \arrow[dd, "\tau"] \\
& \Delta_p \arrow[r, "s_i"'] & X & \\
s_i^{-1}(\mathscr{W}) \arrow[ur] & & & \mathscr{W} \arrow[ul]
\end{tikzcd}$$

Supplementary references

by Claude Sabbah

I have indicated some bibliographical references which appeared after the first publication of the texts in this volume, in order to draw attention to the numerous extensions and applications of the subject. I will use the abbreviation ITSLS to refer to the first text (Introduction to a topological study of Landau singularities), and ISSIH for the second (Introduction to the study of singular integrals and hyperfunctions).

De Rham's book "Variétés différentiables" remains a good reference for the first two chapters of ITSLS, which it covers in greater depth, and there now exists a translation of it into English [dR84]. The reader who wishes to become better acquainted with the theory of complex analytic sets could consult the books [GR65] and [Fis76], as well as the articles in the volume [Vit90].

Cohomological methods have been used in various forms, and notably the methods of sheaf cohomology as recounted in Roger Godement's book [God64]. This theory enables one to give a satisfactory presentation of de Rham's theorem, which can also be found in recent work such as [Voi02] or [Mor01]. With the impetus of the theory of hyperfunctions, a "microlocal" theory of sheaves and their cohomology was developed by Masaki Kashiwara and Pierre Schapira [KS90].

The article of Pierre Dolbeault in [Vit90] summarizes different pieces of work concerning Jean Leray's theory of residues. It must be noted that the version of Leray's theorem used in ITSLS has also been generalized for multivalued differential forms and for (multivalued) forms with poles along an arbitrary complex hypersurface (the comparison theorem due to Alexandre Grothendieck [Gro66] and Pierre Deligne [Del70]).

The most substantial progress has probably been made in the area covered in chapter 4 of ITSLS:

- René Thom's isotopy theorems [Tho69], which had already been used in [PFFL65], were completely proved by John Mather. Other proofs followed

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later, in particular, the one by Jean-Louis Verdier [Ver76]. The proceedings of the conference at Liverpool held in 1970 [Wal71a, Wal71b] gives a clear indication of the considerable activity that has gone on in this area. The book by Mark Goresky and Robert MacPherson [GM88], as well as giving historical pointers and having a very complete bibliography, will also give the reader a precise idea of the results in the domain of the theory of stratified sets. It is interesting to note that the theory of stratified sets now incorporates a microlocal approach (see the book [KS90]) and its influence extends to the theory of linear partial differential equations. The two texts ITSLS and ISSIH collected in this volume are precursors to this unification.

- The theory of singularities, based upon the ideas of Thom, underwent a period of rapid development between 1970 and 1980. In addition to the works of René Thom, which are gathered together in [Tho02], one must mention the work of Heisuke Hironaka, which exerted considerable influence on the French school of singularity theory (see the volume [Pha73b]); also, the work of Vladimir Arnold's school in Moscow, part of which is summarized in [AVGS86]; and, of course, the work of Frédéric Pham himself [Pha67, Pha71b, Pha71a, Pha73a, Pha80a, Pha80b, Pha83b, Pha83c, Pha85, Pha87b]. Finally, John Milnor's book [Mil68] opened the way for a topological study of isolated singularities.

The Picard-Lefschetz formulae occupy a central place in the domain of the topology of complex algebraic varieties, as explained in [Del73] and [Lam81] (see also [Voi02]). They led to a study of the intersection form on the space of the vanishing cycles of a singularity, and are at the foundation of the relation between singularity theory and the theory of semi-simple Lie algebras. The reader can consult [AVGS86] and the references therein, as well as [Vas02] for numerous applications to questions where functions with integral representations appear.

The text ISSIH gives an elementary introduction to the theory of hyperfunctions, for which the basic reference remains [SKK73], which is now conveniently supplemented by [KKK86]. One could also consult the volume [Pha75a], as well as the works [Mor93] and [Kan88].

(Micro)functions in the Nilsson class are presented in the work of F. Pham [Pha75b, Pha76-77, Pha77, Pha83a], and also, for example, in [And92].

The book [KKK86] (in particular its Chapter III) remains a fundamental reference for understanding the importance of microlocal analysis for the study of Feynman integrals. The article [IS69] contributed as an important bridge between mathematicians and physicists in these questions. The development of the application of these techniques to the theory of S-matrices and Feynman integrals is reflected in the articles [Las68, Pha68, Reg68, Reg71, BP80, Bro81, Iag81, KS82, BP83, BD86]. The volume [RIMS76-77] contains fundamental articles on the subject, such as [KK76-77] (see also [KK76, KK77a]), [KS76-77] and [Pha76-77].

Investigating the singularity structure of the Feynman integral by making use of the systems of micro-differential equations that it satisfies has led to important results [KK77b, KKO78, KK79]. The necessary tools from the theory of D-modules can be found in [Kas95], and in more recent books as [Bjö79, Pha80b, Kas03, HTT08], supplemented by [KK81].

Resummation questions (as the resummation of renormalized perturbation series in quantum electrodynamics) has led F. Pham to consider such questions in relation with M. Sato's approach. A survey can be found in [Pha88b] (see also [Pha87a, Pha88a]), which is expanded in the book [CNP93].

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